Commutators involving $\hat{L}_x, \hat{L}_y, \hat{L}_z$ & $\hat{\vec{L}}^2$

Given the commutation relations between the linear momentum and Cartesian coordinates

$$\left[x, \hat{p}_x\right] = \left[y, \hat{p}_y\right] = \left[z, \hat{p}_z\right] = i\hbar$$

We can evaluate the various commutators involving the components of the orbital angular momenta, $\hat{L}_\alpha$. So

$$\left[\hat{L}_x, \hat{L}_y\right] = \left[y\hat{p}_z - z\hat{p}_y, z\hat{p}_x - x\hat{p}_z\right] = \left[y\hat{p}_z, z\hat{p}_x\right] - \left[y\hat{p}_z, x\hat{p}_z\right] - \left[z\hat{p}_y, z\hat{p}_x\right] + \left[z\hat{p}_y, x\hat{p}_z\right]$$

Since $\hat{p}_z$ commutes with x & y we have

$$\left[\hat{L}_x, \hat{L}_y\right] = \left[y\hat{p}_z, z\hat{p}_x\right] + \left[z\hat{p}_y, x\hat{p}_z\right]$$

These are easily evaluated using commutator algebra developed elsewhere in these notes. For example

$$\left[y\hat{p}_z, z\hat{p}_x\right] = y\left[\hat{p}_z, z\hat{p}_x\right] + \left[y, z\hat{p}_x\right]\hat{p}_z = y\left[\hat{p}_z, z\hat{p}_x\right] = y\left[\hat{p}_z, z\right]\hat{p}_x = -i\hbar y\hat{p}_x$$

And in a similar fashion

$$\left[z\hat{p}_y, x\hat{p}_z\right] = \left[z, x\hat{p}_z\right]\hat{p}_y = x\left[z, \hat{p}_z\right]\hat{p}_y = i\hbar x\hat{p}_y$$

And so

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\left(x\hat{p}_y - y\hat{p}_x\right) = i\hbar\hat{L}_z$$

And by symmetry

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x \;\; \& \;\; \left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$$

These commutators are often written using the Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma}$ as

$$\left[\hat{L}_\alpha, \hat{L}_\beta\right] = i\hbar\varepsilon_{\alpha\beta\gamma}\hat{L}_\gamma$$

Where one sums over repeated Greek indices and

$\varepsilon_{\alpha\beta\gamma} = 1$ if $\alpha\beta$ & γ are a cyclic permutation of xy&z

-1 if its an acyclic permutation

0 if two indices are equal.

This tensor is discussed in some detail in the mathematical preliminaries of these notes.

Now for the square of the angular momentum.

$$\hat{\hat{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

and its commutation relations with the individual components.

Consider for example

$$\left[\hat{\hat{L}}^2, \hat{L}_x\right] = \left[\hat{L}_y^2, \hat{L}_x\right] + \left[\hat{L}_z^2, \hat{L}_x\right] = \hat{L}_y\left[\hat{L}_y, \hat{L}_x\right] + \left[\hat{L}_y, \hat{L}_x\right]\hat{L}_y + \hat{L}_z\left[\hat{L}_z, \hat{L}_x\right] + \left[\hat{L}_z, \hat{L}_x\right]\hat{L}_z$$

So

$$\left[\hat{\hat{L}}^2, \hat{L}_x\right] = -i\hbar\hat{L}_y\hat{L}_z - i\hbar\hat{L}_z\hat{L}_y + i\hbar\hat{L}_z\hat{L}_y + i\hbar\hat{L}_y\hat{L}_z = 0$$

So by symmetry

$$\left[\hat{\hat{L}}^2, \hat{L}_y\right] = \left[\hat{\hat{L}}^2, \hat{L}_z\right] = 0$$

Because the individual components of the angular momentum all commute with $\hat{\hat{L}}^2$ but not with one another we can assume that the eigenfunctions of $\hat{H}$ are eigenfunctions of $\hat{\hat{L}}^2$ and one of the three components. Convention has us select $\hat{L}_z$ as the component that shares eigenfunctions with $\hat{\hat{L}}^2$ and $\hat{H}$.