***Electrostatic Potential and Electric Field due to a Collection of Point Charges.***

A charge *Q* at the origin generates and electrostatic potential $\phi(r)$ at a point *r* given by

$$\phi(r) = \kappa \frac{Q}{r}$$

and an electric field $\vec{F}$ given by the negative gradient of the potential,

$$\vec{F} = -\nabla\phi = \kappa \frac{Q}{r^3}\vec{r}$$

where *r* is the magnitude of $\vec{r}$

$$r = \left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$$

$\nabla$ is the Laplacian

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

and $\kappa$ is a constant that depends on the units used for charge and distance. In the SI system, *r* is in meters (M), charge is in Coulombs (C), $\phi$ is in joules (J/C) or Volts, $\vec{F}$ is in Volts/Meter (J/CM) and $\kappa$ is given by

$$\kappa = \frac{1}{4\pi\varepsilon_0} = 0.8987551788x10^{+10}\,JMC^{-2}$$

where the permittivity of free space $\varepsilon_0 = 8.854187816x10^{-12}C^2J^{-1}M^{-1}$. In the SI system the charge on a proton is 1.60217733 $10^{-19}$ C. In the cgs (centimeter-gram-second) system the charge is in esu units, the distance in centimeters, $\phi$ is in ergs/esu, $\vec{F}$ is in stat-volts (ergs/esu-cm) and $\kappa$ is equal to 1.0. In the cgs system the charge on a proton is
4.8029 $10^{-10}$ esu. In the atomic unit system charge is measured in multiples of the proton charge (e), length in multiples of the Bohr radius ($a_0$), $\phi$ in Hartrees (H/charge), $\vec{F}$ in H/charge-length) and $\kappa$ is equal to 1.0. The proton charge in the atomic unit system is 1.

Because of the potential generated by $Q$ a charge q at the point $r$ has a potential energy $V$ given by

$$V = q\phi(r) = \kappa\frac{qQ}{r}$$

In most of our work we will use atomic units. In this system electrostatic potential is measured in multiples of the potential one au distant from a proton. So the unit of potential is

$$\phi(au) = \frac{e}{4\pi\varepsilon_0 a_0} = \frac{1.60217733\cdot 10^{-19}C}{1.12650056\cdot 10^{-10}C^2 J^{-1} M^{-1} 0.5291772\cdot 10^{-10}M} = 27.2113986 J/C$$

Note that a *J/C* is a *Volt*. Electric fields in atomic units are measured in multiples of the field at a distance $a_0$ from a proton, so the atomic unit of electric field is

$$F(au) = \frac{e}{4\pi\varepsilon_0 a_0^2} = \frac{1.60217733\cdot 10^{-19}C}{1.12650056\cdot 10^{-10}C^2 J^{-1} M^{-1}\left(0.5291772\cdot 10^{-10}M\right)^2} = 5.142209\cdot 10^{11} V/M$$

*Exercise:*

*Calculate the potential energy in the SI, cgs, and atomic unit system of a proton and an electron separated by 1Angstrom. Deduce from these results the conversion factors among the three systems. The Bohr radius is equal to 0.5291772 Angstroms.*

If the charge $Q$ was not at the origin but at the terminus of vector $\vec{r}_1$ the electrostatic potential at the point $\vec{r}$ is

$$\phi(\vec{r}) = \frac{Q}{\left|\vec{r} - \vec{r}_1\right|}$$

with the associated electric field

$$\vec{F} = \frac{Q(\vec{r} - \vec{r}_1)}{\left|\vec{r} - \vec{r}_1\right|^3}$$

where the distance between the field point $\vec{r}$ and the charged particle is

$$\left|\vec{r} - \vec{r}_1\right| = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$

*Exercise*

*Given* $\phi(\vec{r}) = \dfrac{Q}{\left|\vec{r} - \vec{r}_1\right|}$ *and* $\vec{F} = -\nabla\phi$ *show that* $\vec{F} = \dfrac{Q(\vec{r} - \vec{r}_1)}{\left|\vec{r} - \vec{r}_1\right|^3}$

The potential due to a collection of N charged particles is the sum of the potential due to each and if the particles are located at the points $\vec{r}_i$ and carry a charge $q_i$ the potential at $\vec{R}$ is given by

$$\phi(\vec{R}) = \sum_{i=1}^{N} \frac{q_i}{\left|\vec{R} - \vec{r}_i\right|}$$

with the associated electric field

$$\vec{F}(\vec{R}) = -\nabla\phi = \sum_{i=1}^{N} \frac{q_i(\vec{R} - \vec{r}_i)}{\left|\vec{R} - \vec{r}_i\right|^3}$$

The potential at a point $\vec{R}$ due to a set of discrete charges satisfies Laplace's equation

$$\nabla^2\phi = 0$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

as one can verify by explicit differentiation. Also because

$$\vec{F} = -\nabla\phi$$

Laplace's equation is equivalent to the divergence of $\vec{F}$ being zero at the point $\vec{R}$

$$\nabla \bullet \vec{F} = 0$$