Wigner or Clebsch-Gordan Coefficients

As we have seen the angular momentum eigenfunction for a composite system can be written as a linear combination of the elements in the direct product space of the vectors of the individual systems. If we combine two systems with angular momentum quantum numbers j_1 & j_2 to form a composite system with angular momentum quantum numbers J & M we have

$$|J,M;j_1,j_2\rangle = \sum_{m_1}\sum_{m_2} |j_1m_1\rangle |j_2m_2\rangle \left(j_1j_2m_1m_2|J,M;j_1,j_2\right)$$

Note that the direct product space is spanned by $(2j_1+1)(2j_2+1)$ orthonormal vectors and the coefficients $\left(j_1j_2m_1m_2|J,M;j_1,j_2\right)$ form a unitary matrix which transforms these vectors into another orthonormal set that are eigenvectors of $\hat{\vec{J}}^2$ & $\hat{J}_z$. These matrix elements are called Wigner or Clebsch-Gordan coefficients and may be determined using powerful group theoretic arguments the discussion of which would take us too far afield but are detailed in several standard texts, such as Messiah, Wigner, and Edmonds.

Let's see how they work.

Assume that $j_2 \le j_1$ and since $M = m_1 + m_2$ we may eliminate the sum over m_1 and write

$$|J,M;j_1,j_2\rangle = \sum_{m_2=-j_2}^{j_2} |j_1,M-m_2\rangle |j_2m_2\rangle \left(j_1j_2M-m_2,m_2|J,M;j_1,j_2\right)$$

The coefficients are collected in tables in the above references and elsewhere (as well as on the internet; see for example; www.volya.net). There is a table for each value of j_2 so if one couples $j_2 = 1/2$ to $j_1 = 1/2, 1, 3/2, 2, \cdots$ one would use the $j_2 = 1/2$ table. If one couples $j_2 = 1$ to $j_1 = 1, 3/2, 2, 5/2, \cdots$ one would use the $j_2 = 1$ table and so on. The table for $j_2 = 1$ is reproduced below.

$$\left(j_1,1,M-m_2,m_2|J,M;j_1,1\right)$$

J	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$

j_1+1	$\sqrt{\frac{(j_1+M)(j_1+M+1)}{(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{(j_1-M+1)(j_1+M+1)}{(2j_1+1)(j_1+1)}}$	$\sqrt{\frac{(j_1-M)(j_1-M+1)}{(2j_1+1)(2j_1+2)}}$
j_1	$-\sqrt{\frac{(j_1+M)(j_1-M+1)}{2j_1(j_1+1)}}$	$\frac{M}{\sqrt{j_1(j_1+1)}}$	$\sqrt{\frac{(j_1-M)(j_1+M+1)}{2j_1(j_1+1)}}$
j_1-1	$\sqrt{\frac{(j_1-M)(j_1-M+1)}{2j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-M)(j_1+M)}{j_1(2j_1+1)}}$	$\sqrt{\frac{(j_1+M)(j_1+M+1)}{2j_1(2j_1+1)}}$

Using this table we can reproduce the various eigenfunctions that we generated when we coupled $j_2=1$ and $j_1=2$. Rewriting this table for $j_1=2$ results in

J	$m_2=1$	$m_2=0$	$m_2=-1$
3	$\sqrt{\frac{(2+M)(3+M)}{30}}$	$\sqrt{\frac{(3-M)(3+M)}{15}}$	$\sqrt{\frac{(2-M)(3-M)}{30}}$
2	$-\sqrt{\frac{(2+M)(3-M)}{12}}$	$\frac{M}{\sqrt{6}}$	$\sqrt{\frac{(2-M)(3+M)}{12}}$
1	$\sqrt{\frac{(2-M)(3-M)}{20}}$	$-\sqrt{\frac{(2-M)(2+M)}{10}}$	$\sqrt{\frac{(2+M)(3+M)}{20}}$

So, for example

$$\left|2M,2,1\right\rangle = -\sqrt{\frac{(2+M)(3-M)}{12}}\left|2,M-1\right\rangle\left|11\right\rangle + \frac{M}{\sqrt{6}}\left|2,M\right\rangle\left|10\right\rangle + \sqrt{\frac{(2-M)(3+M)}{12}}\left|2,M+1\right\rangle\left|1,-1\right\rangle$$

From which

$$\left|22,2,1\right\rangle = -\sqrt{\frac{1}{3}}\left|2,1\right\rangle\left|11\right\rangle + \sqrt{\frac{2}{3}}\left|2,2\right\rangle\left|10\right\rangle$$

That we derived earlier (up to a sign).

References

1. A. Messiah, “Quantum Mechanics” Dover Publications

2. A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press 1974.

3 E. Condon, and G. Shortley, *The Theory of Atomic Spectra*, Cambridge, 1935.

4. E. Wigner