Coupling of Angular Momenta

The previous discussion where we constructed the eigenfunctions and eigenvalues for a spin ½ particle in a spherically symmetric potential is a specific example of a more general problem of finding the eigenfunctions and eigenvalues of the total angular momentum for a system of two independent particles, A and B which now consider.

Let $\left| j_A m_A \right\rangle$ represent the eigenfunctions of $\hat{\vec{j}}_A^2$ & $\hat{j}_{Az}$ with eigenvalues $j_A(j_A+1)$ & m_A respectively with a similar definition for $\left| j_B m_B \right\rangle$ and then define the total angular momentum operator as $\hat{\vec{J}} = \hat{\vec{j}}_A + \hat{\vec{j}}_B$ where we take $\hbar = 1$ for convenience.

We will first show that the total angular momentum operator $\hat{\vec{J}} = \hat{\vec{j}}_A + \hat{\vec{j}}_B$ obeys the usual commutation relations. We then show that $\left[\hat{\vec{J}}^2, \hat{j}_A^2\right] = \left[\hat{\vec{J}}^2, \hat{j}_B^2\right] = 0$, from which we deduce that the eigenfunctions of $\hat{\vec{J}}^2$ & $\hat{J}_z$ are also eigenfunctions of $\hat{j}_A^2$ & $\hat{j}_B^2$. We then describe a technique for expressing these eigenfunctions in terms of those of the particles *A & B*. So let's begin and look at $\left[\hat{J}_\alpha, \hat{J}_\beta\right]$.

Since $\left[\hat{J}_\alpha, \hat{J}_\beta\right] = \left[\hat{j}_{A\alpha} + \hat{j}_{B\alpha}, \hat{j}_{A\beta} + \hat{j}_{B\beta}\right] = \left[\hat{j}_{A\alpha}, \hat{j}_{A\beta}\right] + \left[\hat{j}_{B\alpha}, \hat{j}_{B\beta}\right] = i\varepsilon_{\alpha\beta\gamma}\hat{J}_\gamma$

it follows that

$$\hat{\vec{J}}^2\left|J,M\right\rangle = J(J+1)\left|J,M\right\rangle \;\; \& \;\; \hat{J}_z\left|J,M\right\rangle = M\left|J,M\right\rangle$$

And our task is to relate $\left|J,M\right\rangle$ to $\left| j_A m_A \right\rangle$ and $\left| j_B m_B \right\rangle$. Note first that because $\hat{\vec{J}}^2 = \left(\hat{\vec{j}}_A + \hat{\vec{j}}_B\right)^2$ we can determine $\left[\hat{\vec{J}}^2, \hat{j}_{A\alpha}\right] = 2i\left(\hat{\vec{j}}_A \times \hat{\vec{j}}_B\right)_\alpha$ for $\alpha = x, y, z$.

Which allows one to write

$$\left[\hat{\vec{J}}^2, \hat{j}_{Az}^2\right] = \hat{j}_{Az}\left[\hat{\vec{J}}^2, \hat{j}_{Az}\right] + \left[\hat{\vec{J}}^2, \hat{j}_{Az}\right]\hat{j}_{Az} = \hat{j}_{Az}2i\left(\hat{\vec{j}}_A \times \hat{\vec{j}}_B\right)_z + 2i\left(\hat{\vec{j}}_A \times \hat{\vec{j}}_B\right)_z\hat{j}_{Az}$$

$$\left[\hat{\vec{J}}^2, \hat{j}_{Az}^2\right] = 2i\left\{\hat{j}_{By}\left(\hat{j}_{Az}\hat{j}_{Ax} + \hat{j}_{Ax}\hat{j}_{Az}\right) - \hat{j}_{Bx}\left(\hat{j}_{Az}\hat{j}_{Ay} + \hat{j}_{Ay}\hat{j}_{Az}\right)\right\}$$

And in a similar fashion we see that

$$\left[\hat{\vec{J}}^2, \hat{j}_{Ay}^2\right] = 2i\left\{-\hat{j}_{Bz}\left(\hat{j}_{Ay}\hat{j}_{Ax} + \hat{j}_{Ax}\hat{j}_{Ay}\right) + \hat{j}_{Bx}\left(\hat{j}_{Ay}\hat{j}_{Az} + \hat{j}_{Az}\hat{j}_{Ay}\right)\right\}$$

$$\left[\hat{\vec{J}}^2, \hat{j}_{Ax}^2\right] = 2i\left\{-\hat{j}_{By}\left(\hat{j}_{Ax}\hat{j}_{Az} + \hat{j}_{Az}\hat{j}_{Ax}\right) + \hat{j}_{Bz}\left(\hat{j}_{Ax}\hat{j}_{Ay} + \hat{j}_{Ay}\hat{j}_{Ax}\right)\right\}$$

Adding these three commutators shows that $\left[\hat{\vec{J}}^2, \hat{\vec{j}}_A^2\right] = 0$ and by symmetry $\left[\hat{\vec{J}}^2, \hat{\vec{j}}_B^2\right] = 0$

which means we can write the eigenfunctions of $\hat{\vec{J}}^2$ & $\hat{J}_z$ as being simultaneously eigenfunctions of $\hat{\vec{j}}_A^2$ & $\hat{\vec{j}}_B^2$, i.e.,

$$\hat{\vec{J}}^2\left|JM_J j_A j_B\right\rangle = J(J+1)\left|JM_J j_A j_B\right\rangle$$
$$\hat{J}_z\left|JM_J j_A j_B\right\rangle = M_J\left|JM_J j_A j_B\right\rangle$$
$$\hat{\vec{j}}_A^2\left|JM_J j_A j_B\right\rangle = j_A(j_A+1)\left|JM_J j_A j_B\right\rangle$$
$$\hat{\vec{j}}_B^2\left|JM_J j_A j_B\right\rangle = j_B(j_B+1)\left|JM_J j_A j_B\right\rangle$$

Since $\hat{J}_z$ is the sum of $\hat{j}_{Az}$ & $\hat{j}_{Bz}$ its eigenfunctions are products of those of $\hat{j}_{Az}$ & $\hat{j}_{Bz}$. These products have the form $\left|j_A m_A\right\rangle\left|j_B m_B\right\rangle$ and there are $(2j_A+1)(2j_B+1)$ of them. The collection of these vectors constitute a direct product space and each is an eigenfunction of $\hat{J}_z$ with an eigenvalue $M = m_A + m_B$. In this space there is one vector for which $m_A = j_A$ & $m_B = j_B$, i.e. $\left|j_A j_A\right\rangle\left|j_B j_B\right\rangle$ and $M = j_A + j_B$. Since M cannot be larger than this, this vector must be $\left|JJj_A j_B\right\rangle = \left|j_A j_A\right\rangle\left|j_B j_B\right\rangle$ with $J = j_A + j_B$. There are $2J+1$ vectors associated with a value of J and we can generate the remaining $2J$ of them by operating on $\left|JJj_A j_B\right\rangle = \left|j_A j_A\right\rangle\left|j_B j_B\right\rangle$ with a lowering operator $\hat{J}_- = \hat{j}_{A-} + \hat{j}_{B-}$.

For example

$$\hat{J}_-\left|JJj_A j_B\right\rangle = (\hat{j}_{A-} + \hat{j}_{B-})\left|j_A j_A\right\rangle\left|j_B j_B\right\rangle = \left|j_B j_B\right\rangle\hat{j}_{A-}\left|j_A j_A\right\rangle + \left|j_A j_A\right\rangle\hat{j}_{B-}\left|j_B j_B\right\rangle$$

And since

$$\hat{j}_{A-}\left|j_A j_A\right\rangle = \sqrt{j_A(j_A+1) - j_A(j_A-1)}\left|j_A j_A - 1\right\rangle = \sqrt{2j_A}\left|j_A j_A - 1\right\rangle$$

We have $\hat{J}_-\left|JJj_A j_B\right\rangle = \sqrt{2j_A}\left|j_B j_B\right\rangle\left|j_A j_A - 1\right\rangle + \sqrt{2j_b}\left|j_A j_A\right\rangle\left|j_B j_B - 1\right\rangle$ and using $\hat{J}_-\left|JJj_A j_B\right\rangle = \sqrt{2J}\left|JJ-1j_A j_B\right\rangle$ we see that

$$|JJ-1 j_A j_B\rangle = \sqrt{\frac{j_A}{J}}|j_B j_B\rangle|j_A j_A-1\rangle + \sqrt{\frac{j_B}{J}}|j_A j_A\rangle|j_B j_B-1\rangle$$

We can lower this and generate $|JJ-2 j_A j_B\rangle$ and so on until we have $|J,-J j_A j_B\rangle$ and have all $2J+1$ components.

However these $2J+1$ components number $2(j_A+j_B)+1$ and there are $(2j_A+1)(2j_B+1)$ vectors in this space so we must be able to construct eigenfunctions for values of *J* other than $J = j_A + j_B$. It turns out that one can construct eigenfunctions for values of *J* ranging from the sum $J = j_A + j_B$ to the absolute value of the difference $J = |j_A - j_B|$. To see how this works lets consider the specific example of $j_A = 2$ & $j_B = 1$. We have a 15 dimensional direct product space spanned by the vectors $|2m_A\rangle|1m_B\rangle$ and we can partition this space into subspaces characterized by a particular value of *M* with the largest value being 3 and the lowest -3 for a total of 7 subspaces characterized by *M*=3,2,1,0,-1,-2,-3. The distribution of the functions in these subspaces is shown in the following table. Note that there can only be one function in each of the $M = \pm 3$ subspaces 2 each in the $M = \pm 2$ subspaces 3 in each of the $M = \pm 1$ subspaces and 3 in the $M = 0$ subspace.

The single function in the *M*=3 subspace must be $|3,3,2,1\rangle = |2,2\rangle|1,1\rangle$ and the remaining $J = 3$ functions can be generated by lowering this.

For example after lowering the *M*=3 function we get the *M*=2 state

$$|3,2,2,1\rangle = \sqrt{\frac{2}{3}}|2,1\rangle|1,1\rangle + \sqrt{\frac{1}{3}}|2,2\rangle|1,0\rangle$$

Lowering this results in

$$|3,1,2,1\rangle = \sqrt{\frac{2}{5}}|2,0\rangle|1,1\rangle + \sqrt{\frac{8}{15}}|2,1\rangle|1,0\rangle + \sqrt{\frac{1}{15}}|2,2\rangle|1,-1\rangle$$

and lowering this gives

$$|3,0,2,1\rangle = \sqrt{\frac{1}{5}}|2,-1\rangle|1,1\rangle + \sqrt{\frac{3}{5}}|2,0\rangle|1,0\rangle + \sqrt{\frac{1}{5}}|2,1\rangle|1,-1\rangle$$

The remaining functions with J=3 may be obtained by further lowering operations or written by symmetry.

$$|3,-1,2,1\rangle = \sqrt{\frac{6}{15}}|2,0\rangle|1,-1\rangle + \sqrt{\frac{8}{15}}|2,-1\rangle|1,0\rangle + \sqrt{\frac{1}{15}}|2,-2\rangle|1,1\rangle$$

$$|3,-2,2,1\rangle = \sqrt{\frac{2}{3}}|2,-1\rangle|1,-1\rangle + \sqrt{\frac{1}{3}}|2,-2\rangle|1,0\rangle$$

$$|3,-3,2,1\rangle = |2,-2\rangle|1,-1\rangle$$

$\lvert M\rvert$	m_A	m_B	$\lvert 2m_A\rangle\lvert 1m_B\rangle$
3	±2	±1	$\lvert 2,\pm2\rangle\lvert 1,\pm1\rangle$
2	±2	0	$\lvert 2,\pm2\rangle\lvert 1,0\rangle$
2	±1	±1	$\lvert 2,\pm1\rangle\lvert 1,\pm1\rangle$
1	±2	∓1	$\lvert 2,\pm2\rangle\lvert 1,\mp1\rangle$
1	±1	0	$\lvert 2,\pm1\rangle\lvert 1,0\rangle$
1	0	±1	$\lvert 2,0\rangle\lvert 1,\pm1\rangle$
0	1	-1	$\lvert 2,1\rangle\lvert 1,-1\rangle$
0	0	0	$\lvert 2,0\rangle\lvert 1,0\rangle$
0	-1	1	$\lvert 2,-1\rangle\lvert 1,+1\rangle$

This takes care of the J=3 eigenvectors but what about the remaining and in particular J=2? We know it must be in the M=2 subspace and therefore has the form

$$|2,2,2,1\rangle = A|2,1\rangle|1,1\rangle + B|2,2\rangle|1,0\rangle$$

Where A & B are to be determined so that this function is normalized and orthogonal to the other function in this subspace, namely

$$|3,2,2,1\rangle = \sqrt{\frac{2}{3}}|2,1\rangle|1,1\rangle + \sqrt{\frac{1}{3}}|2,2\rangle|1,0\rangle$$

These constraints require $A^2 + B^2 = 1$ & $A\sqrt{\frac{2}{3}} + B\sqrt{\frac{1}{3}} = 0$ and so

$$|2,2,2,1\rangle = \sqrt{\frac{1}{3}}|2,1\rangle|1,1\rangle - \sqrt{\frac{2}{3}}|2,2\rangle|1,0\rangle$$

And the remaining J=2 vectors may be found by subsequent lowering operations.

$$|2,1,2,1\rangle = \sqrt{\frac{1}{2}}|2,0\rangle|1,1\rangle - \sqrt{\frac{1}{6}}|2,1\rangle|1,0\rangle - \sqrt{\frac{1}{3}}|2,2\rangle|1,-1\rangle$$

$$|2,0,2,1\rangle = \sqrt{\frac{1}{2}}|2,-1\rangle|1,1\rangle - \sqrt{\frac{1}{2}}|2,1\rangle|1,-1\rangle$$

$$|2,-1,2,1\rangle = \sqrt{\frac{1}{2}}|2,0\rangle|1,-1\rangle - \sqrt{\frac{1}{6}}|2,-1\rangle|1,0\rangle - \sqrt{\frac{1}{3}}|2,-2\rangle|1,1\rangle$$

$$|2,-2,2,1\rangle = \sqrt{\frac{1}{3}}|2,-1\rangle|1,-1\rangle - \sqrt{\frac{2}{3}}|2,-2\rangle|1,0\rangle$$

The remaining vectors are those with J=1 i.e., $|1,1,2,1\rangle, |1,0,2,1\rangle \,\&\, |1,-1,2,1\rangle$.

The vector $|1,1,2,1\rangle$ must be in the M=1 subspace and therefore is the linear combination

$$|1,1,2,1\rangle = A|20\rangle|11\rangle + B|21\rangle|10\rangle + C|22\rangle|1-1\rangle$$

$A, B \,\&\, C$ are to be determined so that this vector is normalized and orthogonal to

$$|3,1,2,1\rangle = \sqrt{\frac{2}{5}}|2,0\rangle|1,1\rangle + \sqrt{\frac{8}{15}}|2,1\rangle|1,0\rangle + \sqrt{\frac{1}{15}}|2,2\rangle|1,-1\rangle$$

and

$$|2,1,2,1\rangle = \sqrt{\frac{1}{2}}|2,0\rangle|1,1\rangle - \sqrt{\frac{1}{6}}|2,1\rangle|1,0\rangle - \sqrt{\frac{1}{3}}|2,2\rangle|1,-1\rangle$$

Which results in

$$|1,1,2,1\rangle = \sqrt{\frac{1}{10}}|20\rangle|11\rangle - \sqrt{\frac{3}{10}}|21\rangle|10\rangle + \sqrt{\frac{6}{10}}|22\rangle|1-1\rangle$$

The remaining two functions are obtained by lowering this

$$|1,0,2,1\rangle = \sqrt{\frac{3}{10}}|2-1\rangle|11\rangle - \sqrt{\frac{4}{10}}|20\rangle|10\rangle + \sqrt{\frac{3}{10}}|21\rangle|1-1\rangle$$

$$\left|1,-1,2,1\right\rangle = \sqrt{\frac{6}{10}}\left|2-2\right\rangle\left|11\right\rangle - \sqrt{\frac{3}{10}}\left|2-1\right\rangle\left|10\right\rangle + \sqrt{\frac{1}{10}}\left|20\right\rangle\left|1-1\right\rangle$$

As this example shows one can write the angular momentum eigenfunctions for the total system in terms of the composite system in a fairly straight forward but somewhat tedious way. There is a very powerful technique for doing the same but with considerably less effort that we will now sketch.