Operator Derivation of Eigenvalues and Eigenfunctions of the Angular Momentum

We found that the square of the square of the orbital angular momentum has the eigenvalues $\ell(\ell + 1)\hbar^2$ while its projection along the z axis is $m\hbar$ where both $\ell$ & $m$ are integers by solving a differential equation. We can show, not only that this result follows from the commutation relationships between $\hat{L}^2$ & $\hat{L}_z$ but that these relationships also permit $l$ to be half integer. It’s a mostly algebraic approach that we now develop.

We are interested in the eigenvalue problems

$$\hat{L} \eta_{\alpha\beta} = \alpha \eta_{\alpha\beta} \text{ & } \hat{L}_z \eta_{\alpha\beta} = \beta \eta_{\alpha\beta}$$

Where we assume temporarily that $\hbar = 1$.

Define the operator

$$\hat{L}_a = \hat{L}_a + i\hat{L}_a \text{ and its adjoint } \hat{L}_a = \hat{L}_a - i\hat{L}_a$$

First evaluate the commutator

$$[\hat{L}_a, \hat{L}_z] = [\hat{L}_a, \hat{L}_z] + i[\hat{L}_z, \hat{L}_z] = -\hat{L}_a$$

Then taking the adjoint of this commutator

$$[\hat{L}_a, \hat{L}_z]^\dagger = [\hat{L}_z, \hat{L}_a] = -\hat{L}_a$$

results in $[\hat{L}_a, \hat{L}_z] = \hat{L}_a$

Now operate with $\hat{L}_+$ on $\hat{L}_z \eta_{\alpha\beta} = \beta \eta_{\alpha\beta}$, and consider the left hand side. From above we have

$$\hat{L}_+ \hat{L}_z \eta_{\alpha\beta} = \beta \hat{L}_+ \eta_{\alpha\beta} \text{ and } \hat{L}_+ \hat{L}_z \eta_{\alpha\beta} = (\hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z) \eta_{\alpha\beta} = \hat{L}_z \hat{L}_+ \eta_{\alpha\beta} - \hat{L}_+ \eta_{\alpha\beta}$$

And equating the two we have

$$\hat{L}_z \hat{L}_+ \eta_{\alpha\beta} - \hat{L}_+ \eta_{\alpha\beta} = \beta \hat{L}_+ \eta_{\alpha\beta}$$

and we see that $\hat{L}_+ \eta_{\alpha\beta}$ is an eigenfunction of $\hat{L}_z$ with an eigenvalue $\beta + 1$

$$\hat{L}_z \hat{L}_+ \eta_{\alpha\beta} = (\beta + 1) \hat{L}_+ \eta_{\alpha\beta}$$

Likewise we can show that $\hat{L}_- \eta_{\alpha\beta}$ is an eigenfunction of $\hat{L}_z$ with an eigenvalue $\beta - 1$.
\[ \hat{L}_+ \hat{L}_- \eta_{\alpha\beta} = (\beta - 1) \hat{L}_- \eta_{\alpha\beta} \]

\( \hat{L}_+ \) & \( \hat{L}_- \) are called raising and lowering operators. Since \( \hat{L}_+ \eta_{\alpha\beta} \) are eigenfunctions of \( \hat{L}_z \)

we must have \( \hat{L}_+ \eta_{\alpha\beta} = C_\pm (\alpha, \beta) \eta_{\alpha\beta \mp 1} \). To determine \( C_\pm (\alpha, \beta) \) we note

\[ \langle \hat{L}_+ \eta_{\alpha\beta} | \hat{L}_- \eta_{\alpha\beta} \rangle = |C_\pm (\alpha, \beta)|^2 = \langle \eta_{\alpha\beta} | \hat{L}_- \hat{L}_+ | \eta_{\alpha\beta} \rangle \]

Where we have used the turn-over rule. It’s straightforward to show that

\[ \hat{L}_- \hat{L}_+ = \hat{E}^2 - \hat{L}_z \mp \hat{L}_z \]

from which we have

\[ |C_\pm (\alpha, \beta)|^2 = \langle \eta_{\alpha\beta} | \hat{E}^2 - \hat{L}_z \mp \hat{L}_z | \eta_{\alpha\beta} \rangle = \alpha - \beta^2 \mp \beta = \alpha - \beta (\beta \pm 1) \]

And so

\[ C_\pm (\alpha, \beta) = e^{i\gamma_\kappa} \sqrt{\alpha - \beta (\beta \pm 1)} \]

where \( e^{i\gamma_\kappa} \) is an arbitrary phase factor. Since

\[ |C_+ (\alpha, \beta)|^2 + |C_-(\alpha, \beta)|^2 \geq 0 \]

we must have \( \alpha - \beta^2 \geq 0 \) which in turn requires

\[ -\sqrt{\alpha} \leq \beta \leq \sqrt{\alpha} \]

So for a given \( \alpha \) there exists a minimum and a maximum in the allowed values of \( \beta \). Let \( \underline{\beta} \) be the minimum and \( \overline{\beta} \) the maximum. Then \( \eta_{\alpha\beta} \) cannot be raised and we must require \( \hat{L}_- \eta_{\alpha\beta} = C(\alpha, \overline{\beta}) \eta_{\alpha\beta_{\overline{\beta}+1}} = 0 \) or \( C_+ (\alpha, \overline{\beta}) = \sqrt{\alpha - \overline{\beta} (\overline{\beta} + 1)} = 0 \).

Likewise \( \eta_{\alpha\beta} \) cannot be lowered and so \( C_- (\alpha, \underline{\beta}) = \sqrt{\alpha - \underline{\beta} (\underline{\beta} - 1)} = 0 \) and so

\[ \alpha = \overline{\beta} (\overline{\beta} + 1) = \underline{\beta} (\underline{\beta} - 1) \]

Expressing \( \overline{\beta} \) in terms of \( \underline{\beta} \) results in \( \overline{\beta} = -\frac{1}{2} \pm (\frac{\beta}{2} - \frac{1}{2}) \) or \( \overline{\beta} = -\beta \) and \( \underline{\beta} = \beta - 1 \). Since by definition \( \underline{\beta} \) is the lowest acceptable value for \( \beta \) the value \( \overline{\beta} = \underline{\beta} - 1 \) must be discarded and so \( \overline{\beta} = -\beta \). Since we go from \( \eta_{\alpha\overline{\beta}} \) to \( \eta_{\alpha\underline{\beta}} \) using \( \hat{L}_- \) and reducing \( \overline{\beta} \) by one unit at time the difference \( \overline{\beta} - \underline{\beta} \) must be equal to an integer, say \( N \). We then can write \( \overline{\beta} - \underline{\beta} = N = 2\overline{\beta} \)

where \( \overline{\beta} \) can be a positive integer or half integer, and since \( \alpha = \overline{\beta} (\overline{\beta} + 1) \) we have \( \alpha = \frac{N}{2} (\frac{N}{2} + 1) = j(j + 1) \) where we let \( j \)

represent \( \overline{\beta} \). Possible values of \( \alpha = 0, \frac{1}{2} (\frac{1}{2} + 1), 1(1 + 1), \frac{3}{2} (\frac{3}{2} + 1), 2(2 + 1), \ldots \)
Convention has us write $\beta = m$ which ranges between $-j$ and $j$, $-j \leq m \leq j$, with successive values differing by 1. In summary (reintroducing the units) we have

$$\hat{L}_x \eta_{jm} = (j+1)\hat{h}\eta_{jm} & \hat{L}_z \eta_{jm} = mh\eta_{jm}$$

$$\hat{L}_z \eta_{jm} = e^{\eta_j} \sqrt{j(j+1) - m(m\pm1)}\eta_{jm\pm1}$$

Note that if we have an explicit representation of $\hat{L}^x$ & $\hat{L}^z$ in terms of the spherical polar coordinates $\theta & \phi$, $\eta_{jm}$ is a spherical harmonic, $Y_j^m(\theta, \phi)$ with $j$ being an integer. We can see why this is so as follows. We have seen that $\hat{L}_x \eta_{jj} = 0$ and since $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ we can use the spherical polar coordinate form for $\hat{L}_x$ & $\hat{L}_y$ derived above to write

$$\hat{L}_x \eta_{jj} = e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) \eta_{jj} = 0$$

If we can find $\eta_{jj}$ we can generate $\eta_{jm}$ using $\hat{L}_-$. If we write $\eta_{jj}(\theta, \phi) = A(\phi)B(\theta)$ we find that

$$\frac{1}{B(\theta)\cot \theta} \frac{dB}{d\theta} + i \frac{1}{A(\phi)} \frac{dA}{d\phi} = 0$$

so if $i \frac{1}{A(\phi)} \frac{dA}{d\phi} = c$ then $\frac{1}{B(\theta)\cot \theta} \frac{dB}{d\theta} = -c$. From these we find

$$A(\phi) = e^{-ic\phi} & B(\theta) = \sin^{-c} \theta$$

$$\eta_{jj}(\theta, \phi) = A(\phi)B(\theta) = N_{jj} e^{-ic\phi} \sin^{-c} \theta$$

Where $N_{jj}$ is a normalization factor. We can determine $c$ by noting that since

$$\hat{L}_z \eta_{jj} = j\hbar \eta_{jj}$$

we have $-i \frac{d\eta_{jj}}{d\phi} = jn_{jj}$ so $j = -c$ and $\eta_{jj}(\theta, \phi) = N_{jj} e^{i\phi} \sin^j \theta$. If we require that $\eta_{jj}(\theta, \phi)$ be single valued $\eta_{jj}(\theta, \phi) = \eta_{jj}(\theta, \phi + 2\pi)$ then, we find that $j$ must be 0 or a positive integer as before and up to a phase factor $\eta_{jj}(\theta, \phi) = Y_j^j(\theta, \phi)$. The half integer quantum numbers for angular momentum are usually associated with the intrinsic angular momentum or spin of a particle. This brings us to the next topic,
constructing the angular momentum wave function for a spin $\frac{1}{2}$ particle moving in a spherically symmetric potential.