Operator Derivation of Eigenvalues and Eigenfunctions of the Angular Momentum

We found that the square of the square of the orbital angular momentum has the eigenvalues $\ell(\ell+1)\hbar^2$ while its projection along the z axis is $m\hbar$ where both ℓ & m are integers by solving a differential equation. We can show, not only that this result follows from the commutation relationships between $\hat{\vec{L}}^2$ & $\hat{L}_z$ but that these relationships also permit *l* to be half integer. It's a mostly algebraic approach that we now develop.

We are interested in the eigenvalue problems

$$\hat{\vec{L}}^2\eta_{\alpha\beta} = \alpha\eta_{\alpha\beta} \ \ \& \ \ \hat{L}_z\eta_{\alpha\beta} = \beta\eta_{\alpha\beta}$$

Where we assume temporarily that $\hbar = 1$.

Define the operator

$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and its adjoint $\hat{L}_+^\dagger = \hat{L}_- = \hat{L}_x - i\hat{L}_y$

First evaluate the commutator

$$\left[\hat{L}_+, \hat{L}_z\right] = \left[\hat{L}_x, \hat{L}_z\right] + i\left[\hat{L}_y, \hat{L}_z\right] = -\hat{L}_+$$

Then taking the adjoint of this commutator

$$\left[\hat{L}_+, \hat{L}_z\right]^\dagger = \left[\hat{L}_z, \hat{L}_-\right] = -\hat{L}_-$$

results in $\left[\hat{L}_-, \hat{L}_z\right] = \hat{L}_-$

Now operate with $\hat{L}_+$ on $\hat{L}_z\eta_{\alpha\beta} = \beta\eta_{\alpha\beta}$ and consider the left hand side. From above we have

$$\hat{L}_+\hat{L}_z\eta_{\alpha\beta} = \beta\hat{L}_+\eta_{\alpha\beta} \ \text{ and } \ \hat{L}_+\hat{L}_z\eta_{\alpha\beta} = \left(\hat{L}_z\hat{L}_+ - \hat{L}_+\right)\eta_{\alpha\beta} = \hat{L}_z\hat{L}_+\eta_{\alpha\beta} - \hat{L}_+\eta_{\alpha\beta}$$

And equating the two we have

$$\hat{L}_z\hat{L}_+\eta_{\alpha\beta} - \hat{L}_+\eta_{\alpha\beta} = \beta\hat{L}_+\eta_{\alpha\beta}$$

and we see that $\hat{L}_+\eta_{\alpha\beta}$ is an eigenfunction of $\hat{L}_z$ with an eigenvalue $\beta+1$

$$\hat{L}_z\hat{L}_+\eta_{\alpha\beta} = (\beta+1)\hat{L}_+\eta_{\alpha\beta}$$

Likewise we can show that $\hat{L}_-\eta_{\alpha\beta}$ is an eigenfunction of $\hat{L}_z$ with an eigenvalue $\beta-1$

$\hat{L}_z \hat{L}_- \eta_{\alpha\beta} = (\beta - 1)\hat{L}_- \eta_{\alpha\beta}$

$\hat{L}_+$ & $\hat{L}_-$ are called raising and lowering operators. Since $\hat{L}_\pm \eta_{\alpha\beta}$ are eigenfunctions of $\hat{L}_z$ we must have $\hat{L}_\pm \eta_{\alpha\beta} = C_\pm(\alpha,\beta)\eta_{\alpha\beta\pm 1}$. To determine $C_\pm(\alpha,\beta)$ we note

$\left\langle \hat{L}_\pm \eta_{\alpha\beta} \middle| \hat{L}_\pm \eta_{\alpha\beta} \right\rangle = \left| C_\pm(\alpha,\beta) \right|^2 = \left\langle \eta_{\alpha\beta} \middle| \hat{L}_\mp \hat{L}_\pm \middle| \eta_{\alpha\beta} \right\rangle$

Where we have used the turn-over rule. It's straightforward to show that

$\hat{L}_\mp \hat{L}_\pm = \hat{\vec{L}}^2 - \hat{L}_z^2 \mp \hat{L}_z$ from which we have

$\left| C_\pm(\alpha,\beta) \right|^2 = \left\langle \eta_{\alpha\beta} \middle| \hat{\vec{L}}^2 - \hat{L}_z^2 \mp \hat{L}_z \middle| \eta_{\alpha\beta} \right\rangle = \alpha - \beta^2 \mp \beta = \alpha - \beta(\beta \pm 1)$

And so

$C_\pm(\alpha,\beta) = e^{i\gamma_\pm}\sqrt{\alpha - \beta(\beta \pm 1)}$ where $e^{i\gamma_\pm}$ is an arbitrary phase factor. Since $\left| C_+(\alpha,\beta) \right|^2 + \left| C_-(\alpha,\beta) \right|^2 \geq 0$ we must have $\alpha - \beta^2 \geq 0$ which in turn requires $-\sqrt{\alpha} \leq \beta \leq \sqrt{\alpha}$. So for a given α there exists a minimum and a maximum in the allowed values of β. Let $\underline{\beta}$ be the minimum and $\bar{\beta}$ the maximum. Then $\eta_{\alpha\bar{\beta}}$ cannot be raised and we must require $\hat{L}_+ \eta_{\alpha\bar{\beta}} = C(\alpha,\bar{\beta})\eta_{\alpha\bar{\beta}+1} = 0$ or $C_+(\alpha,\bar{\beta}) = \sqrt{\alpha - \bar{\beta}(\bar{\beta}+1)} = 0$. Likewise $\eta_{\alpha\underline{\beta}}$ cannot be lowered and so $C_-(\alpha,\bar{\beta}) = \sqrt{\alpha - \underline{\beta}(\underline{\beta}-1)} = 0$ and so $\alpha = \bar{\beta}(\bar{\beta}+1) = \underline{\beta}(\underline{\beta}-1)$. Expressing $\bar{\beta}$ in terms of $\underline{\beta}$ results in $\bar{\beta} = -\frac{1}{2} \pm (\underline{\beta} - \frac{1}{2})$ or $\bar{\beta} = -\underline{\beta}$ and $\bar{\beta} = \underline{\beta} - 1$. Since by definition $\underline{\beta}$ is the lowest acceptable value for β the value $\bar{\beta} = \underline{\beta} - 1$must be discarded and so $\bar{\beta} = -\underline{\beta}$. Since we go from $\eta_{\alpha\bar{\beta}}$ to $\eta_{\alpha\underline{\beta}}$ using $\hat{L}_-$ and reducing $\bar{\beta}$ by one unit at time the difference $\bar{\beta} - \underline{\beta}$ must be equal to an integer, say N. We then can write $\bar{\beta} - \underline{\beta} = N = 2\bar{\beta}$ where $\bar{\beta}$ can be a positive integer or half integer, and since $\alpha = \bar{\beta}(\bar{\beta}+1)$ we have $\alpha = \frac{N}{2}(\frac{N}{2}+1) = j(j+1)$ where we let j represent $\bar{\beta}$. Possible values of $\alpha = 0,\ \frac{1}{2}(\frac{1}{2}+1),\ 1(1+1),\ \frac{3}{2}(\frac{3}{2}+1),\ 2(2+1),\cdots$

Convention has us write $\beta = m$ which ranges between $-j$ and j, $-j \le m \le j$, with successive values differing by 1. In summary (reintroducing the units) we have

$$\hat{\vec{L}}^2 \eta_{jm} = j(j+1)\hbar^2 \eta_{jm} \;\; \& \;\; \hat{L}_z \eta_{jm} = m\hbar \eta_{jm}$$

$$\hat{L}_\pm \eta_{jm} = e^{i\gamma_\pm} \sqrt{j(j+1) - m(m\pm 1)}\,\eta_{jm\pm 1}$$

Note that if we have an explicit representation of $\hat{\vec{L}}^2$ & $\hat{L}_z$ in terms of the spherical polar coordinates $\theta \,\&\, \phi$, η_{jm} is a spherical harmonic, $Y_j^m(\theta,\phi)$ with j being an integer. We can see why this is so as follows. We have seen that $\hat{L}_+ \eta_{jj} = 0$ and since $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ we can use the spherical polar coordinate form for $\hat{L}_x \,\&\, \hat{L}_y$ derived above to write

$$\hat{L}_+ \eta_{jj} = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)\eta_{jj}(\theta,\phi) = 0$$

If we can find η_{jj} we can generate η_{jm} using $\hat{L}_-$. If we write $\eta_{jj}(\theta,\phi) = A(\phi)B(\theta)$ we find that

$$\frac{1}{B(\theta)\cot\theta}\frac{dB}{d\theta} + i\frac{1}{A(\phi)}\frac{dA}{d\phi} = 0$$

so if $i\dfrac{1}{A(\phi)}\dfrac{dA}{d\phi} = c$ then $\dfrac{1}{B(\theta)\cot\theta}\dfrac{dB}{d\theta} = -c$. From these we find

$A(\phi) = e^{-ic\phi}$ & $B(\theta) = \sin^{-c}\theta$ or

$$\eta_{jj}(\theta,\phi) = A(\phi)B(\theta) = N_{jj} e^{-ic\phi}\sin^{-c}\theta$$

Where N_{jj} is a normalization factor. We can determine c by noting that since

$\hat{L}_z \eta_{jj} = j\hbar \eta_{jj}$ we have $-i\dfrac{d\eta_{jj}}{d\phi} = j\eta_{jj}$ so $j = -c$ and $\eta_{jj}(\theta,\phi) = N_{jj} e^{ij\phi}\sin^j\theta$. If we require that $\eta_{jj}(\theta,\phi)$ be single valued $\eta_{jj}(\theta,\phi) = \eta_{jj}(\theta,\phi + 2\pi)$ then, we find that j must be 0 or a positive integer as before and up to a phase factor $\eta_{jj}(\theta,\phi) = Y_j^j(\theta,\phi)$.

The half integer quantum numbers for angular momentum are usually associated with the intrinsic angular momentum or spin of a particle. This brings us to the next topic, constructing the angular momentum wave function for a spin ½ particle moving in a spherically symmetric potential.