Angular momentum in a central potential

The Hamiltonian for a particle moving in a spherically symmetric potential is

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + V(r)$$

and if $\hat{\vec{L}}$ is to be constant we must have

$$\left[\hat{H}, \hat{\vec{L}}\right] = 0$$

So let's evaluate this commutator. In what follows we will make frequent use of the commutator relationship

$$\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$$

Since $\hat{\vec{L}}$ is a vector we need to consider the three components and we start with $\hat{L}_x$.

$$\left[\hat{H}, \hat{L}_x\right] = \left[\frac{\hat{\vec{p}}^2}{2m} + V(r), \hat{L}_x\right] = \frac{1}{2m}\left[\hat{\vec{p}}^2, \hat{L}_x\right] + \left[V(r), \hat{L}_x\right]$$

$$\left[\hat{\vec{p}}^2, \hat{L}_x\right] = \left[\hat{p}_x^2, \hat{L}_x\right] + \left[\hat{p}_y^2, \hat{L}_x\right] + \left[\hat{p}_z^2, \hat{L}_x\right]$$

Consider the first term $\left[\hat{p}_x^2, \hat{L}_x\right]$ and note

$$\left[\hat{p}_x^2, \hat{L}_x\right] = \hat{p}_x\left[\hat{p}_x, \hat{L}_x\right] + \left[\hat{p}_x, \hat{L}_x\right]\hat{p}_x$$

And since $\left[\hat{p}_x, \hat{L}_x\right] = \left[\hat{p}_x, y\hat{p}_z - z\hat{p}_y\right] = 0$

$$\left[\hat{p}_x^2, \hat{L}_x\right] = 0$$

The next term is $\left[\hat{p}_y^2, \hat{L}_x\right]$ which we evaluate as

$$\left[\hat{p}_y^2, \hat{L}_x\right] = \hat{p}_y\left[\hat{p}_y, \hat{L}_x\right] + \left[\hat{p}_y, \hat{L}_x\right]\hat{p}_y$$

And since $\left[\hat{p}_y, \hat{L}_x\right] = \left[\hat{p}_y, y\hat{p}_z - \hat{p}_z y\right] = \left[\hat{p}_y, y\right]\hat{p}_z - \hat{p}_z\left[\hat{p}_y, y\right] = 0$

we have $\left[\hat{p}_y^2, \hat{L}_x\right] = 0$.

In a similar way we can show that

$\left[\hat{p}_z^2, \hat{L}_x\right] = 0$ and so $\left[\hat{\vec{p}}^2, \hat{L}_x\right] = 0$

Since there is nothing special about the x direction it follows that

$$\left[\hat{\vec{p}}^2, \hat{L}_x\right] = \left[\hat{\vec{p}}^2, \hat{L}_y\right] = \left[\hat{\vec{p}}^2, \hat{L}_z\right] = 0$$

The remaining term to be considered is $\left[V(r), \hat{\vec{L}}\right]$ so let's begin with $\left[V(r), \hat{L}_x\right]$

$$\left[V(r), \hat{L}_x\right] = \left[V(r), y\hat{p}_z - z\hat{p}_y\right] = y\left[V(r), \hat{p}_z\right] - z\left[V(r), \hat{p}_y\right]$$

Since

$$\left[V(r), \hat{p}_z\right] = i\hbar\frac{\partial V}{\partial z} = i\hbar\frac{z}{r}\frac{\partial V}{\partial r}$$

and

$$\left[V(r), \hat{p}_y\right] = i\hbar\frac{\partial V}{\partial y} = i\hbar\frac{y}{r}\frac{\partial V}{\partial r}$$

We have

$$\left[V(r), \hat{L}_x\right] = i\hbar\frac{\partial V}{\partial r}\left(yz - zy\right) = 0$$

And by symmetry

$$\left[V(r), \hat{L}_y\right] = \left[V(r), \hat{L}_z\right] = 0$$

So $\left[\hat{H}, \hat{\vec{L}}\right] = 0$, as anticipated. Note that since $\hat{\vec{L}}$ commutes with $\hat{H}$, $\hat{\vec{L}}^2$ also commutes.

We then have four operators that commute with $\hat{H}$ but what are the commutation relations among these four?