Spin Operators in Many Electron Systems

In an N electron system the spin angular momentum operators are formed by summing the operators for the individual spins as follows:

$$\hat{S}_\mu = \sum_{j=1}^{N} \hat{s}_\mu(j)$$

where $\mu = x, y, z$

and $\hat{s}_\mu(j)$ is the μ^{th} component of the spin operator for the j^{th} electron. Because the spin operators for different electrons commute the commutation relations for the many-electron operators are then

$$\left[\hat{S}_\mu, \hat{S}_\nu\right] = i\varepsilon_{\mu\nu\gamma}\hat{S}_\gamma$$

The operator representing the square of the total spin angular momentum is

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$$

Which may be written as

$$\hat{S}^2 = \hat{S}_\pm \hat{S}_\mp \mp \hat{S}_z + \hat{S}_z^2$$

Where the many electron raising and lowering operators are

$$\hat{S}_\pm = \sum_{j=1}^{N} \hat{s}_\pm(j)$$

Exercise

Prove that $\hat{S}^2 = \hat{S}_\pm \hat{S}_\mp \mp \hat{S}_z + \hat{S}_z^2$

Note that $\hat{S}^2$ commutes with the spin free Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \hat{f}(i) + \sum_{i<j}^{N} g(i,j)$$

$\left[\hat{S}^2, \hat{H}\right] = 0$ and each of its components $\left[\hat{S}^2, \hat{S}_\mu\right] = 0$ but because the individual components do not commute we may choose the eigenfunctions of $\hat{H}$ to be eigenfunctions of $\hat{S}^2$ and one of its components (most often chosen to be the z component) so the many electron eigenfunction $\psi(1,2,3,\cdots N)$ of satisfies the equations

$$\hat{S}^2\psi = S(S+1)\psi \;\; \& \;\; \hat{S}_z\psi = M\psi$$

where $-S \le M \le S$ in multiples of 1.