Harmonic Oscillator in a Constant Electric Field

Consider a one dimensional harmonic oscillator in a constant electric field $\vec{F}$, and let the charge on the oscillator be q. If the oscillator is on the x axis, the Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2 + q\phi(x)$$

In one dimension

$F\hat{x} = -\dfrac{d\phi}{dx}\hat{x}$ and since the field is constant this integrates to $\phi(x) = \phi(0) - Fx \equiv -Fx$ where we will neglect the constant $\phi(0)$ which simply shifts the zero of energy. We then seek the eigenvalues and eigenfunctions of he Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2 - qFx$$

We can solve this problem exactly and then compare the perturbation and exact results.

First note that we can complete the square on the potential

$$\frac{1}{2}kx^2 - qFx = \frac{k}{2}\left(x^2 - \frac{2qF}{k}x\right) = \frac{k}{2}\left(\left(x - \frac{qF}{k}\right)^2 - \left(\frac{qF}{k}\right)^2\right)$$

and then define the new variable $\xi = x - \dfrac{qF}{k}$, resulting in the Schrodinger equation

$$\hat{H}\Phi(\xi) = \left(-\frac{\hbar^2}{2m}\frac{d^2}{d\xi^2} + \frac{1}{2}k\xi^2 + \frac{(qF)^2}{2k}\right)\Phi(\xi) = E\Phi(\xi)$$

or

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{d\xi^2} + \frac{1}{2}k\xi^2\right)\Phi(\xi) = \left(E + \frac{(qF)^2}{2k}\right)\Phi(\xi)$$

This is the harmonic oscillator equation, so, as we have seen above

$$\Phi_n(\xi) = N_n H_n(\alpha^{1/2}\xi)e^{-\alpha\xi^2} \text{ and } E_n = h\nu(n+\tfrac{1}{2}) - \frac{(qF)^2}{2k}$$

Note that all of the levels have been lowered by the same amount and the wavefunctions have all been shifted along the x axis so that they are centered at $x = \frac{qF}{k}$. If q and F are both positive the equilibrium point is shifted in the $+x$ direction as expected.
The explicit dependence of the first two wavefunctions on the electric field is

$$\Phi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha\left(x-\frac{qF}{k}\right)^2/2}$$

$$\Phi_1(x) = \left(\frac{4\alpha^3}{\pi}\right)^{1/4}\left(x - \frac{qF}{k}\right)e^{-\alpha\left(x-\frac{qF}{k}\right)^2/2}$$

Now let's use perturbation theory to solve the problem.

The perturbation is $-qFx$ and the first order correction to the energy is zero by parity. The second order correction is then

$$E_n^{(2)} = \sum_{p\neq n}^{\infty} \frac{\left|\langle p|-qFx|n\rangle\right|^2}{h\nu(n-p)} = \frac{(qF)^2}{h\nu}\sum_{p\neq n}^{\infty}\frac{(x)_{pn}^2}{(n-p)}$$

From the above only terms that will appear in the summation are $p = n\pm 1$ so we have

$$E_n^{(2)} = \frac{(qF)^2}{h\nu}\left(\frac{(x)_{n,n-1}^2}{1} + \frac{(x)_{n.n+1}^2}{-1}\right)$$

$$(x)_{n,n-1}^2 = \frac{n}{2\alpha} \;\; \& \;\; (x)_{n,n+1}^2 = \frac{n+1}{2\alpha}$$

so

$$E_n^{(2)} = -\frac{(qF)^2}{2k}$$

Note that the third order correction to the energy has the form

$E_n^{(3)} = \sum_{p\neq n}\sum_{t\neq n}\frac{x_{np}x_{pt}x_{tn}}{E_{np}^0 E_{nt}^0}$ which is identically zero. Indeed all higher corrections must be zero and the exact energy of the oscillator in the field is

$E_n = h\nu(n+\tfrac{1}{2}) - \frac{(qF)^2}{2k}$ as found in the exact solution. In this problem the second order correction is the total correction. The correction to the wavefunction is given by

$$\mathbf{\Phi}_n^{(1)} = \frac{-qF}{h\nu}\sum_{p\neq n}\frac{(x)_{np}\,\mathbf{\Phi}_p^0}{n-p} = \frac{-qF}{h\nu}\left(\frac{(x)_{n,n-1}\,\mathbf{\Phi}_{n-1}^0}{1} + \frac{(x)_{n,n+1}\,\mathbf{\Phi}_{n+1}^0}{-1}\right) = \frac{-qF}{\sqrt{2\alpha}h\nu}\left(\sqrt{n}\mathbf{\Phi}_{n-1}^0 - \sqrt{n+1}\mathbf{\Phi}_{n+1}^0\right)$$