Expectation Value

Since $\psi^*(x)\,\psi(x)\,dx$ is a measure of the probability of finding the particle in the interval dx around the point x, the average value of x is given by

$$\int x\,\psi^*(x)\,\psi(x)\,dx = \langle x \rangle$$

where $\langle x \rangle$ is called the expectation value of x.

For a particle on a line

$$\psi = \psi_N(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{N\pi x}{a}\right)$$

So

$$\langle x \rangle = \frac{2}{a}\int_0^a x\,\sin^2\left(\frac{N\pi x}{a}\right)dx$$

Let $z = \frac{N\pi x}{a}$; $dx = \frac{a}{N\pi}\,dz$ & $x = \frac{a}{N\pi}\,z$

$$\langle x \rangle = \frac{2}{a}\left(\frac{a}{N\pi}\right)^2 \int_0^{N\pi} z\,\sin^2 z\,dz$$

The standard integral

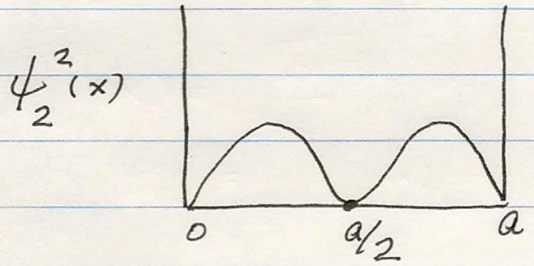
$$\int z\,\sin^2 z\,dz = \frac{z^2}{4} - \frac{z\,\sin 2z}{4} - \frac{\cos 2z}{8}$$

results in

$$\int_0^{N\pi} z \sin^2 z \, dz = \frac{(N\pi)^2}{4}$$

so $$\langle x \rangle = \frac{2}{a} \frac{a^2}{(N\pi)^2} \frac{(N\pi)^2}{4} = \frac{a}{2} \; ; \; \underline{\underline{\text{for all } N}}$$

& The ~~most~~ average location of the particle is in the center of the "box". Reconcile this with



where in this state we are unlikely to find the particle at $\frac{a}{2}$

Momentum Operator

The kinetic energy of a particle classically is

$$\frac{1}{2} M v^2 = \frac{1}{2M} (Mv)^2 = \frac{P^2}{2M}$$

where P is the linear momentum. In quantum mechanics the kinetic energy operator is

$$-\frac{\hbar^2}{2M} \frac{d^2}{dx^2}$$ & if we associate this with $\frac{P^2}{2M}$

we make the correspondence

$$\hat{p}^2 = -\hbar^2 \frac{d^2}{dx^2} = -\left(\hbar \frac{d}{dx}\right)\left(\hbar \frac{d}{dx}\right)$$

& this suggests that

$$\hat{P}_x = i\hbar \frac{d}{dx}$$

The general rule for calculating the expectation value of any operator is.

$$\langle \hat{T} \rangle = \int \psi^* \hat{T} \psi \, dx$$

i.e., we sandwhich it between ψ^* & ψ, let it operate on ψ & then do the integration. Note that

$$\langle x \rangle = \int \psi^*(x)\, x\, \psi(x)\, dx = \int x\, \psi^*(x)\, \psi(x)\, dx$$

since x is only a multiplicative operator it doesn't matter where in the integrand we place it.

It does make a difference for $\hat{P}_x$

$$\langle \hat{P}_x \rangle = \int \psi^*(x) \frac{\hbar}{i}\left(\frac{d}{dx}\psi\right) dx = \frac{\hbar}{i} \int \psi^*(x) \left(\frac{d\psi}{dx}\right) dx$$

For the particle on a line (in a box)

$$\langle P_x \rangle = \frac{\hbar}{i} \frac{2}{a} \int_0^a \sin\left(\frac{N\pi x}{a}\right) \frac{d}{dx} \sin\left(\frac{N\pi x}{a}\right) dx$$

$$= \frac{\hbar}{i} \frac{2}{a} \frac{N\pi}{a} \int_0^a \sin\left(\frac{N\pi x}{a}\right) \cos\left(\frac{N\pi x}{a}\right) dx$$

Let $z = \frac{N\pi x}{a}$; $x = \frac{a}{N\pi} z$ $\quad dx = \frac{a}{N\pi} dz$

$$\langle P_x \rangle = \frac{\hbar}{i} \frac{2N\pi}{a^2} \frac{a}{N\pi} \int_0^{N\pi} \sin z \cos z \, dz$$

using $\int \sin z \cos z \, dz = \frac{1}{2} \sin^2 z$

we see $\langle P_x \rangle \equiv 0$

We recall that the momentum is a vector & this result suggests that if we measure the momentum of the particle many times we are equally likely to find it moving to the left as to the right.