Bohr Model of the Atom

The Hydrogen atom consists of a relatively massive positively charged (*e*) nucleus (proton) around which an electron of charge –*e* moves.
The force attracting the electron to the proton is given by Coulombs law

$$\vec{f} = \frac{e^2 \hat{\vec{r}}}{4\pi\varepsilon_0 r^2}$$

$r$ is the magnitude of the position vector from the proton to the electron and $\varepsilon_0 = 8.85419x10^{-12} C^2 N^{-1} m^{-2}$ is the permittivity of free space.

The velocity of the electron is equal to the time derivative of the position vector $\vec{V} = \frac{d\vec{r}}{dt}$ and if the orbit is circular

$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

where $\hat{i}$ & $\hat{j}$ are unit vectors. The velocity becomes

$$\vec{V} = \frac{d\vec{r}}{dt} = r\frac{d\theta}{dt}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)$$

and the magnitude is

$$V = r\frac{d\theta}{dt} = r\omega$$

We need the acceleration which is

$$\frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = r\frac{d^2\theta}{dt^2}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) - r\left(\frac{d\theta}{dt}\right)^2\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

Identifying

$$\hat{\vec{r}} = \left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

means that the counterbalancing force is

$$rm\left(\frac{d\theta}{dt}\right)^2 = rm\omega^2 = \frac{m\mathrm{V}^2}{r}$$

Equating the two forces gives

$$\frac{e^2}{4\pi\varepsilon_0 r^2} = rm\omega^2 = \frac{m\mathrm{V}^2}{r}$$

Or

$$\frac{e^2 m}{4\pi\varepsilon_0 r} = (m\mathrm{V})^2 = p^2 = \frac{\hbar^2 n^2}{r^2}$$

Where Bohr used the de Broglie relationship $p = \frac{h}{\lambda}$ and assumed that $2\pi r = n\lambda \quad n = 1, 2, 3, .$ This results in the radius of the electrons orbit being

$$r = \frac{\hbar^2 n^2 4\pi\varepsilon_0}{e^2 m} = a_0 n^2 \quad n = 1, 2, 3, \cdots$$

Where $a_0 = \frac{\hbar^2 4\pi\varepsilon_0}{e^2 m}$ is equal to $0.529177x10^{-10}m$. Note that it is the smallest radius possible for the electron ($n = 1$) and is called the Bohr radius. One often refers to $n = 1$ as the first Bohr orbit, $n = 2$ as the second Bohr orbit and so on.

The energy of the electron in the H atom is the sum of its kinetic ($T$) and potential energies ($V$)

$$E = T + V = \frac{1}{2}m\mathrm{V}^2 - \frac{e^2}{4\pi\varepsilon_0 r} = -\frac{e^2}{8\pi\varepsilon_0 r}$$

And using the value of the radius found above the energy becomes

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2 \hbar^2 n^2} = \frac{E_1}{n^2} \quad n = 1, 2, 3, \cdots$$

Where

$$E_1 = -\frac{me^4}{2(4\pi\varepsilon_0)^2 \hbar^2} = -2.17987x10^{-18} J = -13.606eV$$

$E_1$ is the lowest possible energy for the H atom. As with the radius of the allowed orbits one refers to $E_1$ as the energy of the electron in the first Bohr orbit, $E_2$ as the energy of the electron in the second Bohr orbit and so on. It's useful to remember the possible energies of the H atom as

$$E_n = -\frac{13.606}{n^2} eV \quad n = 1, 2, 3, \cdots$$

How fast is the electron moving in the allowed orbits? We can estimate this as follows

Since

$$2\pi r = n\lambda = \frac{nh}{p}$$

We have

$$p = \frac{nh}{2\pi r} = \frac{nh}{2\pi a_0 n^2} = m\mathrm{v}$$

And so

$$\frac{h}{2\pi a_0 mn} = \frac{\hbar}{a_0 mn} = \mathrm{v}$$

So the speed (magnitude of the velocity) is $\mathrm{v} = \frac{\hbar}{a_0 mn} = \frac{2.1877x10^6}{n} m/s$

It's insightful to express this speed relative to the speed of light $c$.

$$\mathrm{v} = \frac{c}{137.0}$$