Time Dependence of Ψ. Section 4.4

The wavefunction of a system $\Psi(x,t)$

evolves in time according to the time dependent SE.

$$\hat{\mathcal{H}}\,\Psi(x,t) = i\hbar \frac{\partial \Psi}{\partial t}$$

(for the particle on a line $\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$)

Most of the time $\hat{\mathcal{H}}$ will not contain (depend) on the time variable t. That lets us find a solution to this time dependent equation using the separation of variables technique.

Let $\Psi(x,t) = \psi(x)\,f(t)$

Then $\hat{\mathcal{H}}\Psi = f(t)\,\hat{\mathcal{H}}\,\psi(x)$

& $\frac{\partial \Psi}{\partial t} = \psi(x)\,\frac{df}{dt}$

so $f(t)\,\hat{\mathcal{H}}\,\psi(x) = i\hbar\,\psi(x)\,\frac{df}{dt}$

which can be written as

$$\frac{1}{\psi(x)} \hat{\mathcal{H}} \psi(x) = i\hbar \frac{1}{f(t)} \frac{df}{dt}$$

Since x & t are independent variables this can only be true if each side equals a constant. Note that the dimensions of $\frac{1}{\psi} \hat{\mathcal{H}} \psi$ are those of $\hat{\mathcal{H}}$ & these are energy, so we call this constant E for energy.

Then we have the two equations

$$\hat{\mathcal{H}} \psi = E \psi \quad (\text{as before!})$$

&

$$\frac{df}{dt} = \frac{E}{i\hbar} f(t) = -\frac{iE}{\hbar} f(t)$$

The time dependent solution is

$$f(t) = e^{-\frac{iE}{\hbar} t} + \text{constant}$$

By convention the constant is taken to be zero.

This has no effect on our subsequent analysis

The solution to the time dependent SE is then

$$\Psi_N(x,t) = \psi_N(x)\, e^{-\frac{iE_N t}{\hbar}}$$

These are called <u>stationary state</u> solutions because

$$\Psi_N^*(x,t)\,\Psi_N(x,t)\,dx = \psi_N^*(x)\, e^{\frac{iE_N t}{\hbar}}\, \psi(x)\, e^{-\frac{iE_N t}{\hbar}}\, dx$$

$$= \psi_N^*(x)\,\psi(x)\,dx$$

& the probability of finding the particle in the interval dx doesn't change with time. These are sometimes called pure quantum states. We will often write the solution as $\psi_N(x)\, e^{-i\omega_N t}$

where $\hbar\omega_N = E_N$.

General Solution to $\hat{\mathcal{H}}\Psi = i\hbar \dfrac{\partial \Psi}{\partial t}$

Note that if we form the linear combination of all possible stationary state solutions

$$\sum_{N=1}^{\infty} c_N \psi_N(x) e^{-i\omega_N t} = \Psi(x,t)$$

this is also a solution to the time dependent SE

Proof:

$$\hat{H}\Psi = \hat{H}\sum_{N=1}^{\infty} c_N \psi_N e^{-i\omega_N t} = \sum_{N=1}^{\infty} c_N E_N \psi_N(x) e^{-i\omega_N t}$$

$$\& \quad \frac{\partial \Psi}{\partial t} = \sum_{N=1}^{\infty} c_N \psi_N(x) \frac{d}{dt} e^{-i\omega_N t} = -\frac{i}{\hbar}\sum_{N=1}^{\infty} c_N E_N \psi_N(x) e^{-i\omega_N t}$$

$$\& \text{ so } \quad \hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t} \qquad \text{QED}$$

We then have an ∞ number of solutions to the time dependent SE depending on how we choose the constants c_N.

Note that

$$\int \Psi^*(x,t)\Psi(x,t)\,dx = \sum_{N=1}^{\infty}\sum_{M=1}^{\infty} c_N^* c_M e^{-i(\omega_M - \omega_N)t} \delta_{NM}$$

$$= \sum_{N=1}^{\infty} |c_N|^2 = 1 \qquad \text{(Normalization)}$$

If the wavefunction for the system is

$$\Psi(x,t) = \sum_{N=1}^{\infty} c_N \psi_N(x)\, e^{-i\omega_N t}$$

then the probability of finding the system in stationary state N is $|c_N|^2$.

Example

Suppose we "prepare" our particle on a line states so that $c_1 = c_2 = \frac{1}{\sqrt{2}}$; $c_N = 0$ for $N \geq 3$

$$\Psi(x,t) = \frac{1}{\sqrt{2}} \psi_1(x)\, e^{-i\omega_1 t} + \frac{1}{\sqrt{2}} \psi_2(x)\, e^{-i\omega_2 t}$$

probability density $\Psi^*(x,t)\,\Psi(x,t)$

$$\Psi(x,t) = \frac{e^{-i\omega_1 t}}{\sqrt{2}} \left(\psi_1(x) + \psi_2(x)\, e^{-i\overbrace{(\omega_2-\omega_1)}^{\Delta\omega} t} \right)$$

$$\Psi^*\Psi = \frac{1}{2}\left(\psi_1^2(x) + \psi_2^2(x) + \psi_1(x)\psi_2(x)\left(e^{-i\Delta\omega t} + e^{+i\Delta\omega t}\right)\right)$$

$$\Psi^*\Psi = \frac{1}{2}\left(\psi_1^2(x) + \psi_2^2(x) + 2\psi_1(x)\psi_2(x)\cos(t\Delta\omega) \right)$$

$$\Delta\omega = \omega_2 - \omega_1 = \frac{E_2 - E_1}{\hbar} = \frac{37.61\,(3)}{a^2\, 0.6582 \times 10^{-15}}$$

$$= \frac{1.71 \times 10^{17}}{a^2}\ s^{-1} \quad \text{for an electron}$$

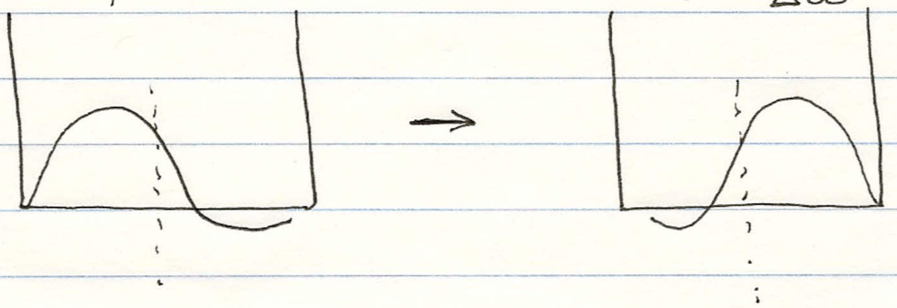
at $t = 0$

$$\Psi^*\Psi = \frac{1}{2}\left(\psi_1 + \psi_2\right)^2$$

at $\quad t\Delta\omega = \pi, \quad t = \dfrac{\pi}{\Delta\omega}$

$$\Psi^*\Psi = \frac{1}{2}\left(\psi_1 - \psi_2\right)^2$$

$t = 0$ $\qquad\qquad$ $t = \dfrac{\pi}{\Delta\omega}$



The particle will oscillate from the left hand side of the line to the right hand side with a frequency $\Delta\omega$.

What is the energy of the particle?

$$\langle E \rangle = \int \Psi^*(x,t)\, \hat{\mathcal{H}}\, \Psi(x,t)$$

$$= \frac{1}{2} \int \left(\psi_1(x) + \psi_2(x) e^{it\Delta\omega} \right) \left(E_1 \psi_1(x) + E_2 \psi_2(x) e^{-it\Delta\omega} \right) dx$$

$$\langle E \rangle = \frac{1}{2}(E_1 + E_2)$$

If we measure the energy many times, half the times we will find E_1 & the other half, E_2. The average is then $\frac{1}{2}(E_1+E_2)$

This illustrates another general principle of QM. The only possible results of the measurement of the energy is one of the eigenvalues of the Hamiltonian operator, E_N.

If the system is in a stationary state ψ_N we will always measure E_N. In a more general non-stationary state we measure E_N with a probability $|C_N|^2$.

Consider the expectation value of x in the two state systems

$$\langle x \rangle = \frac{1}{2}\int x\left(\psi_1^2 + \psi_2^2 + 2\psi_1\psi_2 \cos t\Delta\omega\right)dx$$

$$= \frac{1}{2}\left(\frac{a}{2} + \frac{a}{2}\right) + \cos t\Delta\omega \int_0^a x\psi_1(x)\psi_2(x)\,dx$$

The integral

$$\int_0^a x\psi_1(x)\psi_2(x)\,dx = \frac{2}{a}\int_0^a x \sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\,dx$$

can be evaluated by noting

$$\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a} = \frac{1}{2}\left\{\cos\frac{\pi x}{a} - \cos\frac{3\pi x}{a}\right\}$$

& using $\int z\cos z\,dz = \cos z + z\sin z$

resulting in

$$\int_0^a x\psi_1(x)\psi_2(x)\,dx = \frac{64a}{9\pi^2}$$

so

$$\langle x \rangle = \frac{a}{2} + \frac{64a}{9\pi^2} \cos t\Delta\omega$$

the speed of the particle can be estimated as

$$\frac{d\langle x \rangle}{dt} = -\frac{64\, a\, \Delta\omega}{9\pi^2} \sin(t\Delta\omega)$$

So the particle stops & turns around every $t = \frac{\pi}{\Delta\omega}$ sec.