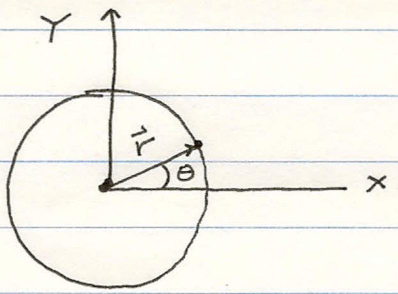
Particle on a circle



Consider a particle constrained to move on a circle of radius r. The SE is

$$-\frac{\hbar^2}{2M}\frac{d^2\psi}{dx^2}-\frac{\hbar^2}{2M}\frac{d^2\psi}{dY^2}=E\,\psi(x,Y)$$

The only thing that is changing is the angle θ so we want to write the derivatives with respect to x and Y in terms of θ.

since $\quad X = r\cos\theta \quad \text{\&} \quad Y = r\sin\theta$

$$\cos\theta = \frac{x}{\sqrt{x^2+Y^2}} \quad \text{\&} \quad \sin\theta = \frac{Y}{\sqrt{x^2+Y^2}}$$

using the chain rule

$$\frac{d\psi}{dx} = \frac{d\psi}{d\theta}\left(\frac{d\theta}{dx}\right) \quad \text{\&} \quad \frac{d\psi}{dY} = \frac{d\psi}{d\theta}\left(\frac{d\theta}{\partial Y}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{r^2} \quad \& \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

Continuing

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial x}\right) = \frac{\partial \psi}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \left(\frac{\partial \theta}{\partial x}\right)^2 \frac{\partial^2 \psi}{\partial \theta^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial y}\right) = \frac{\partial \psi}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2} + \left(\frac{\partial \theta}{\partial y}\right)^2 \frac{\partial^2 \psi}{\partial \theta^2}$$

Since $\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{r^4}$ & $\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{r^4}$

we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$$

& the Schrödinger equation becomes

$$\frac{-\hbar^2}{2Mr^2}\frac{\partial^2 \psi}{\partial \theta^2} = E\,\psi(\theta)$$

$$\frac{\partial^2 \psi}{\partial \theta^2} = -\frac{2Mr^2 E}{\hbar^2}\psi$$

$Mr^2 = I$, the moment of inertia

so $$\frac{\partial^2 \psi}{\partial \theta^2} = -\frac{2IE}{\hbar^2}\psi = -\omega^2 \psi(\theta)$$

where $$E = \frac{(\hbar\omega)^2}{2I}$$

$$\psi = A e^{-i\omega\theta}$$

Boundary condition $\psi(\theta) = \psi(\theta + 2\pi)$

$$e^{-i\omega\theta} = e^{-i\omega\theta} e^{-i\omega 2\pi}$$

requires that $\omega = 0, \pm 1, \pm 2, \cdots$ integers

call these integers M

$$\psi = A e^{-iM\theta}$$

fix A by normalization. We require

$$\int_0^{2\pi} \psi^*(\theta)\, \psi(\theta)\, d\theta = 1$$

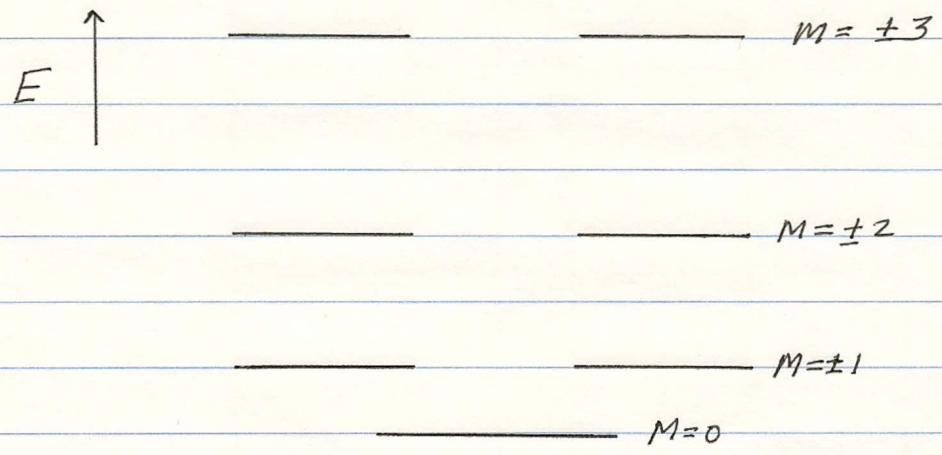
so $A^2 2\pi = 1$ & $A = \dfrac{1}{\sqrt{2\pi}}$

$$\therefore \quad \psi_M(\theta) = \frac{1}{\sqrt{2\pi}} e^{-iM\theta} \qquad M = 0, \pm 1, \pm 2, \cdots$$

The energy levels in this system are

$$E_M = \frac{\hbar^2 M^2}{2I} \;; \quad M = 0, \pm 1, \pm 2, \ldots$$

Note that the lowest energy is non degenerate & equals 0, while the others are each doubly degenerate



The probability of finding the particle in the interval $d\theta$ around the point θ is given by

$$\psi_M^*(\theta)\,\psi_M(\theta)\,d\theta = \frac{1}{2\pi} e^{iM\theta} e^{-iM\theta}\,d\theta = \frac{d\theta}{2\pi}$$

& there is a uniform probability of finding the particle anywhere along the circle.

The linear momentum of this particle is a vector with a x & y component

$$\hat{\bar{P}} = \hat{x}\frac{\hbar}{i}\frac{\partial}{\partial x} + \hat{y}\frac{\hbar}{i}\frac{\partial}{\partial y}$$

since $\quad \dfrac{\partial \psi}{\partial x} = \dfrac{\partial \psi}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \dfrac{-y}{r^2}\dfrac{\partial \psi}{\partial \theta}$

& $\quad \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \psi}{\partial \theta}\dfrac{\partial \theta}{\partial y} = \dfrac{+x}{r^2}\dfrac{\partial \psi}{\partial \theta}$

we can write this operator as

$$\hat{\bar{P}} = \frac{\hbar}{ir^2}\left(-y\hat{x} + x\hat{y}\right)\frac{\partial}{\partial \theta}$$

recall that the angular momentum is given by

$\bar{l} = \vec{r} \times \bar{P}$ so, the angular momentum operator is

$$\hat{\bar{l}} = \frac{\hbar}{ir^2}\,\bar{r} \times \left(-y\hat{x} + x\hat{y}\right)\frac{\partial}{\partial \theta}$$

& $\vec{r} \times \left(-y\hat{x} + x\hat{y}\right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & 0 \\ -y & x & 0 \end{vmatrix}$

so $\vec{r} \times (-y\hat{x} + x\hat{y}) = (x^2 + y^2)\hat{z} = r^2\hat{z}$

& so

$$\hat{\ell} = \hat{z}\,\frac{\hbar}{i}\,\frac{\partial}{\partial\theta}$$

& $\frac{\hbar}{i}\frac{\partial}{\partial\theta} = \hat{\ell}_z$ (the z component of the angular momentum)

note the expectation value of $\hat{\ell}_z$

$$\langle \hat{\ell}_z \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{+iM\theta} \cdot \frac{\hbar}{i}\frac{\partial}{\partial\theta} e^{-iM\theta}\, d\theta$$

$$= \frac{\hbar}{2\pi i}(-iM)\,2\pi = -M\hbar \text{ or } M\hbar \text{ since}$$

$M = 0, \pm 1, \pm 2, \ldots$