CEM 993 **Spring 2023**

Assignment No. 2 (20 pts.) **Due Date: February 10, 2023**

1. (3 pts.) Use Wick's theorem to write the expressions (in terms of the n-products with or without contractions) for the operator product $M_1 M_2 M_3 M_4 M_5$, where M_i, $i = 1-5$, are the creation or annihilation operators.

2. (4 pts.) Use Wick's theorem to show that

$$X_p X_q X_r^\dagger X_s^\dagger = X_r^\dagger X_s^\dagger X_p X_q + \delta_{pr} X_s^\dagger X_q - \delta_{ps} X_r^\dagger X_q - \delta_{qr} X_s^\dagger X_p + \delta_{qs} X_r^\dagger X_p + \delta_{ps}\delta_{qr} - \delta_{pr}\delta_{qs}. \quad (1)$$

   Verify the correctness of Eq. (1) by applying the anticommutation relations for the creation and annihilation operators.

3. (6 pts.) Use Wick's theorem and rules of calculating the expectation values of normal products $n[\ldots]$ with or without contractions relative to the true vacuum to evaluate the following matrix element:

$$\langle\{p_1 p_2\}|Z|\{q_1 q_2\}\rangle, \quad (2)$$

   where

$$Z = \sum_{r,s} \langle r|z|s\rangle X_r^\dagger X_s \quad (3)$$

   is a one-body operator and

$$|\{p_1 p_2\}\rangle = X_{p_1}^\dagger X_{p_2}^\dagger |0\rangle \quad (4)$$

   and

$$|\{q_1 q_2\}\rangle = X_{q_1}^\dagger X_{q_2}^\dagger |0\rangle \quad (5)$$

   are the 2-fermion Slater determinants (as usual, $|0\rangle$ designates the true vacuum state). Do not use the general Slater rules for matrix elements involving one-body operators derived in class, but, rather, calculate the matrix element given by Eq. (2) directly by applying to it the rules of determining the expectation values of the products of the creation or annihilation operators in the true vacuum state, resulting from Wick's theorem. Once you are done, demonstrate that the resulting expression for the matrix element given by Eq. (2) recovers the Slater rules for the two cases that lead the nonzero results, i.e., $\langle\{p_1 p_2\}|Z|\{p_1 p_2\}\rangle$ and $\langle\{p_1 p_2\}|Z|\{q_1 p_2\}\rangle$.

4. (3 pts.) Calculate the following N-products:

   (a) $N[X_a X_b X_i^\dagger X_j^\dagger]$,

   (b) $N[X_i X_j X_a^\dagger X_b^\dagger]$,

   (c) $N[X_i X_a X_j^\dagger X_b^\dagger]$.

5. (4 pts.) Let $\chi(p)$ and $\pi(p)$ be the "step" functions defined as follows:

$$\chi(p) = \begin{cases} 1, & \text{for } p = i \\ 0, & \text{for } p = a \end{cases} , \quad \pi(p) = \begin{cases} 0, & \text{for } p = i \\ 1, & \text{for } p = a \end{cases} . \tag{6}$$

Show that

(a) $\overbracket{Y_p\, X_q} = \chi(q)\delta_{pq}$,

(b) $\overbracket{Y_p\, X_q^\dagger} = \pi(q)\delta_{pq}$,

(c) $\overbracket{Y_p^\dagger\, X_q} = 0$,

(d) $\overbracket{Y_p^\dagger\, X_q^\dagger} = 0$.