CEM 993 **Spring 2023**

Assignment No. 1 (20 pts.) **Due Date: January 27, 2023**

1. (4 pts.) Let the operator $\mathcal{A}$ be defined as follows:

$$\mathcal{A} = \frac{1}{N!} \sum_{R \in S_N} (-1)^R R, \tag{1}$$

where $R \in S_N$ are permutations belonging to a group of all permutations of N objects, S_N. Prove the following properties of $\mathcal{A}$:

   (a) $\mathcal{A}R = (-1)^R \mathcal{A}$, for every $R \in S_N$,

   (b) $\mathcal{A}^2 = \mathcal{A}$ [using (a)].

2. (4 pts.) Use an analysis similar to that carried out in one of our classes to prove the following anti-commutation relation:

$$\{X_p, X_q\} = 0, \tag{2}$$

where X_p is the annihilation operator associated with the single-particle state $|p\rangle$. Use Eq. (2) to show that

$$\{X_p^\dagger, X_q^\dagger\} = 0, \tag{3}$$

where $X_p^\dagger$ is the creation operator associated with the single-particle state $|p\rangle$.

3. (3 pts.) Prove the following relationships:

$$[X_p, N_q] = \delta_{pq} X_p \tag{4}$$

and

$$[X_p^\dagger, N_q] = -\delta_{pq} X_p^\dagger, \tag{5}$$

where X_p, $X_p^\dagger$, and N_p are, respectively, the annihilation, creation, and number operators associated with the single-particle state $|p\rangle$.

4. (3 pts.) Use the anticommutation rules for the creation and annihilation operators to demonstrate that the total number operator,

$$N = \sum_r X_r^\dagger X_r, \tag{6}$$

commutes with the one-body operator

$$Z = \sum_{p,q} \langle p|z|q\rangle X_p^\dagger X_q. \tag{7}$$

5. (3 pts.) Let $|\{pq\}\rangle = X_p^\dagger X_q^\dagger|0\rangle$ and $|\{rst\}\rangle = X_r^\dagger X_s^\dagger X_t^\dagger|0\rangle$ be the configurations belonging to the two- and three-particle Hilbert spaces, $\mathcal{H}_2$ and $\mathcal{H}_3$, respectively. We use the usual notation, where $X_p^\dagger$ is the creation operator associated with the single-particle state $|p\rangle$ and $|0\rangle$ is the vacuum state. Use the anti-commutation relations for the creation and annihilation operators to show that configurations $|\{pq\}\rangle$ and $|\{rst\}\rangle$ are orthogonal.

6. (3pts.) Calculate the following normal products without or with contractions:

   (a) $n[X_pX_qX_r^\dagger X_s^\dagger]$,

   (b) $n[X_pX_q^\dagger X_rX_s^\dagger]$,

   (c) $n[X_pX_qX_r^\dagger X_s^\dagger]$,

   (d) $n[X_pX_qX_r^\dagger X_s^\dagger]$,

   (e) $n[X_pX_qX_r^\dagger X_s^\dagger]$,

   (f) $n[X_pX_qX_r^\dagger X_s^\dagger]$.

SOLUTIONS

1. (a)

We must prove that

$$\mathcal{A} R = (-1)^R \mathcal{A}$$

for any permutation $R \in \mathcal{S}_N$.

Using the definition of $\mathcal{A}$, we obtain

$$\mathcal{A} R = \frac{1}{N!}\left[\sum_{P \in \mathcal{S}_N} (-1)^P P\right] R$$

$$= \frac{1}{N!} \sum_{P \in \mathcal{S}_N} (-1)^P PR .$$

Let us substitute $PR = S \in \mathcal{S}_N$, so that $P = SR^{-1}$. When P runs over the entire group $\mathcal{S}_N$, $PR = S$ runs over the entire group, too.

Thus,

$$\mathcal{A}R = \frac{1}{N!}\sum_{P\in S_N}(-1)^P PR = \frac{1}{N!}\sum_{S\in S_N}(-1)^{SR^{-1}} S.$$

Now, the sign of permutation SR^{-1} equals

$$(-1)^{SR^{-1}} = (-1)^S(-1)^{R^{-1}} = (-1)^S(-1)^R,$$

since the sign of the inverse permutation R^{-1} is the same as the sign of R, i.e., $(-1)^{R^{-1}} = (-1)^R$.

As a result,

$$\mathcal{A}R = \frac{1}{N!}\sum_{S\in S_N}(-1)^S(-1)^R S$$

$$= (-1)^R \underbrace{\frac{1}{N!}\sum_{S\in S_N}(-1)^S S}_{\mathcal{A}} = (-1)^R\mathcal{A}.$$

(b) Let us apply the relationship $\mathcal{A}R = (-1)^R\mathcal{A}$ to calculate $\mathcal{A}^2$.

We obtain,

$$\mathcal{A}^2 = \mathcal{A}\mathcal{A} = \mathcal{A}\left[\frac{1}{N!}\sum_{R\in S_N}(-1)^R R\right] = \frac{1}{N!}\sum_{R\in S_N}(-1)^R \mathcal{A} R$$
$$= \frac{1}{N!}\sum_{R\in S_N}(-1)^R(-1)^R\mathcal{A},$$ so that

$$\mathcal{A}^2 = \frac{1}{N!}\sum_{R\in S_N}\mathcal{A} = \frac{1}{N!}N!\,\mathcal{A} = \mathcal{A},$$

There are $N!$ elements R here

which completes the proof.

2. There are two cases:

(i) $p \neq q$; (ii) $p = q$.

We begin with (i):

We can assume that $p < q$ (the analysis of the $p > q$ case is very similar).

There are four different types of basis states $|\ldots n_p \ldots n_q \ldots\rangle$ that we must consider:

- $n_p = 0$, $n_q = 0$. In this case,

$$\{X_p, X_q\} | \cdots \cancel{p} \cdots \cancel{q} \cdots \rangle =$$
$$= X_p X_q | \cdots \cancel{p} \cdots \cancel{q} \cdots \rangle + X_q X_p | \cdots \cancel{p} \cdots \cancel{q} \cdots \rangle$$
$$= 0 + 0 = 0.$$

- $n_p = 0$, $n_q = 1$. In this case,

$$\{X_p, X_q\} | \cdots \cancel{p} \cdots q \cdots \rangle =$$
$$= X_p X_q | \overbrace{\cdots}^{m} \cancel{p} \overbrace{\cdots}^{n} q \cdots \rangle + X_q X_p | \overbrace{\cdots}^{m} \cancel{p} \overbrace{\cdots}^{n} q \cdots \rangle$$
$$= (-1)^{m+n} X_p | \cdots \cancel{p} \cdots \cancel{q} \cdots \rangle + 0$$
$$= 0,$$ where we used the notation in which m is the number of occupied levels preceding p and n is the number of occupied levels between p and q.

- $n_p = 1$, $n_q = 0$. In this case,

$$\{X_p, X_q\} | \cdots p \cdots \cancel{q} \cdots \rangle =$$
$$= X_p X_q | \cdots p \cdots \cancel{q} \cdots \rangle + X_q X_p | \overbrace{\cdots}^{m} p \overbrace{\cdots}^{n} \cancel{q} \cdots \rangle$$

$$= 0 + (-1)^m X_q |\cdots \cancel{p} \cdots \cancel{q} \cdots\rangle$$

$= 0$ (again, m is the number of occupied levels preceding p).

- $n_p = 1$, $n_q = 1$. In this case,

$$\{X_p, X_q\} |\cdots p \cdots q \cdots\rangle =$$

$$= X_p X_q |\overset{m}{\widetilde{\cdots}}\, p \overset{n}{\widetilde{\cdots}}\, q \cdots\rangle + X_q X_p |\overset{m}{\widetilde{\cdots}}\, p \overset{n}{\widetilde{\cdots}}\, q \cdots\rangle$$

$$= (-1)^{m+n+1} X_p |\overset{m}{\widetilde{\cdots}}\, p \cdots \cancel{q} \cdots\rangle$$

$$+ (-1)^m X_q |\overset{m}{\widetilde{\cdots}}\, \cancel{p} \overset{n}{\widetilde{\cdots}}\, q \cdots\rangle$$

$$= (-1)^{m+n+1} (-1)^m |\cdots \cancel{p} \cdots \cancel{q} \cdots\rangle$$

$$+ (-1)^m (-1)^{m+n} |\cdots \cancel{p} \cdots \cancel{q} \cdots\rangle$$

$$= (-1)^{n+1} |\cdots \cancel{p} \cdots \cancel{q} \cdots\rangle + (-1)^n |\cdots \cancel{p} \cdots \cancel{q} \cdots\rangle$$

$$= [(-1) + 1] (-1)^n |\cdots \cancel{p} \cdots \cancel{q} \cdots\rangle = 0,$$

where m is the number of occupied levels preceding p and n is the number of occupied levels between p and q.

In all four cases,

$$\{X_p, X_q\} |\ldots\rangle = 0. \text{ Thus,}$$

$$\{X_p, X_q\} = 0 \quad \text{for } p \neq q \ (p > q).$$

(ii) $p = q$.

There are two different types of situations in this case:

- $n_p = 0$. In this case,

$$\{X_p, X_p\} |\ldots \cancel{p} \ldots\rangle$$
$$= 2 X_p X_p |\ldots \cancel{p} \ldots\rangle = 0.$$

- $n_p = 1$. In this case,

$$\{X_p, X_p\} |\ldots p \ldots\rangle =$$
$$= 2 X_p X_p |\overbrace{\ldots}^{m} p \ldots\rangle =$$
$$= 2 (-1)^m X_p |\ldots \cancel{p} \ldots\rangle = 0,$$

where m is the number of levels preceding p.

In either situation,

$$\{X_p, X_q\} = 0.$$

Thus, $\{X_p, X_q\} = 0$ for all p and q.

Based on the above relation, we can write,

$$\{X_p^\dagger, X_q^\dagger\} = X_p^\dagger X_q^\dagger + X_q^\dagger X_p^\dagger$$

$$= (X_q X_p)^\dagger + (X_p X_q)^\dagger$$

$$= (X_p X_q + X_q X_p)^\dagger$$

$$= \{X_p, X_q\}^\dagger = 0$$ for all p and q.

3. $$[X_p, N_q] = X_p X_q^\dagger X_q - X_q^\dagger X_q X_p$$
$$= (\delta_{pq} - X_q^\dagger X_p) X_q + X_q^\dagger X_p X_q$$
$$= \delta_{pq} X_q - \cancel{X_q^\dagger X_p X_q} + \cancel{X_q^\dagger X_p X_q}$$
$$= \delta_{pq} X_q = \delta_{pq} X_p.$$

$$[X_p^\dagger, N_q] = X_p^\dagger N_q - N_q X_p^\dagger$$
$$= (N_q^\dagger X_p)^\dagger - (X_p N_q^\dagger)^\dagger$$
$$= (N_q X_p)^\dagger - (X_p N_q)^\dagger$$
$$= (N_q X_p - X_p N_q)^\dagger$$
$$= -[X_p, N_q]^\dagger = -\delta_{pq} X_p^\dagger,$$

where we used the fact that

$$N_q^\dagger = (X_q^\dagger X_q)^\dagger = X_q^\dagger X_q = N_q.$$

4. We must calculate

$$[N, Z] = NZ - ZN$$

and demonstrate that the result is zero.

$$[N, Z] = NZ - ZN$$

$$= \left(\sum_r X_r^+ X_r\right)\left(\sum_{pq} \langle p|\hat{z}|q\rangle X_p^+ X_q\right)$$

$$- \left(\sum_{pq} \langle p|\hat{z}|q\rangle X_p^+ X_q\right)\left(\sum_r X_r^+ X_r\right)$$

$$= \sum_{pqr} \langle p|\hat{z}|q\rangle X_r^+ X_r X_p^+ X_q$$

$$- \sum_{pqr} \langle p|\hat{z}|q\rangle X_p^+ X_q X_r^+ X_r .$$

We obtain,

$$[N, Z] =$$

$$= \sum_{pqr} \langle p|z|q\rangle X_r^{\dagger} X_r X_p^{\dagger} X_q$$

$$- \sum_{pqr} \langle p|z|q\rangle X_p^{\dagger} X_q X_r^{\dagger} X_r$$

$$= \sum_{pqr} \langle p|z|q\rangle X_r^{\dagger} (\delta_{rp} - X_p^{\dagger} X_r) X_q$$

$$- \sum_{pqr} \langle p|z|q\rangle X_p^{\dagger} (\delta_{qr} - X_r^{\dagger} X_q) X_r$$

$$= \sum_{pqr} \langle p|z|q\rangle \delta_{rp} X_r^{\dagger} X_q$$

$$- \sum_{pqr} \langle p|z|q\rangle X_r^{\dagger} X_p^{\dagger} X_r X_q$$

$$- \sum_{pqr} \langle p|z|q\rangle \delta_{qr} X_p^{\dagger} X_r$$

$$+ \sum_{pqr} \langle p|z|q\rangle X_p^{\dagger} X_r^{\dagger} X_q X_r$$

$$= \sum_{pq} \langle p|z|q\rangle X_p^{\dagger} X_q \; -$$

$$
\begin{aligned}
& - \sum_{pqr} \langle p | \hat{z} | q \rangle X_r^\dagger X_p^\dagger X_r X_q \\
& - \sum_{pq} \langle p | \hat{z} | q \rangle X_p^\dagger X_q \\
& + \sum_{pqr} \langle p | \hat{z} | q \rangle X_r^\dagger X_p^\dagger X_r X_q \\
& = 0 .
\end{aligned}
$$

5. $$\langle \{pq\} | \{rst\} \rangle = \langle 0 | X_q X_p X_r^\dagger X_s^\dagger X_t^\dagger | 0 \rangle$$

$$= \langle 0 | X_q (\delta_{pr} - X_r^\dagger X_p) X_s^\dagger X_t^\dagger | 0 \rangle$$

$$= \delta_{pr} \langle 0 | X_q X_s^\dagger X_t^\dagger | 0 \rangle$$

$$- \langle 0 | X_q X_r^\dagger X_p X_s^\dagger X_t^\dagger | 0 \rangle$$

$$= \delta_{pr} \langle 0 | (\delta_{qs} - X_s^\dagger X_q) X_t^\dagger | 0 \rangle$$

$$- \langle 0 | (\delta_{qr} - X_r^\dagger X_q) X_p X_s^\dagger X_t^\dagger | 0 \rangle$$

$$= \delta_{pr} \delta_{qs} \langle 0 | X_t^\dagger | 0 \rangle$$

$$- \delta_{pr} \langle 0 | X_s^\dagger X_q X_t^\dagger | 0 \rangle$$

$$- \delta_{qr} \langle 0 | X_p X_s^\dagger X_t^\dagger | 0 \rangle$$

$$+ \langle 0| X_r^\dagger X_q X_p X_s^\dagger X_t^\dagger |0\rangle .$$

Since $X_p|0\rangle = 0$, so that $\langle 0|X_p^\dagger = 0$,

we can write

$$\langle\{pq\}|\{rst\}\rangle = -\delta_{qr}\langle 0| X_p X_s^\dagger X_t^\dagger |0\rangle$$

$$= -\delta_{qr}\langle 0|(\delta_{ps} - X_s^\dagger X_p) X_t^\dagger |0\rangle$$

$$= -\delta_{qr}\delta_{ps}\langle 0| X_t^\dagger |0\rangle$$

$$+ \delta_{qr}\langle 0| X_s^\dagger X_p X_t^\dagger |0\rangle = 0,$$

since again $\langle 0|X_p^\dagger = 0$ for every p.

We just proved that $\langle\{pq\}|\{rst\}\rangle = 0$,

i.e., $|\{pq\}\rangle$ and $|\{rst\}\rangle$ are orthogonal.

6. (a) $n[X_p X_q X_r^\dagger X_s^\dagger] = X_r^\dagger X_s^\dagger X_p X_q$.

(b) $n[X_p X_q^\dagger X_r X_s^\dagger] = - X_q^\dagger X_s^\dagger X_p X_r$.

(c) $n[X_p X_q X_r^\dagger X_s^\dagger] = X_p X_q \, n[X_r^\dagger X_s^\dagger]$

$= 0$, since $X_p X_q = 0$.

(d) $n[X_p X_q X_r^\dagger X_s^\dagger] = X_p X_s^\dagger \, n[X_q X_r^\dagger]$

$= - \delta_{ps} X_r^\dagger X_q$.

(e) $n[X_p X_q X_r^\dagger X_s^\dagger] = - X_p X_r^\dagger X_q X_s^\dagger$

$= - \delta_{pr} \delta_{qs}$ (we used $n[\emptyset]=1$).

(f) $n[X_p X_q X_r^\dagger X_s^\dagger] = X_p X_s^\dagger X_q X_r^\dagger$

$= \delta_{ps} \delta_{qr}$ (again, we used $n[\emptyset]=1$).