CEM 993 **Spring 2023**

Assignment No. 1 (20 pts.) **Due Date: January 27, 2023**

1. (4 pts.) Let the operator $\mathcal{A}$ be defined as follows:

$$\mathcal{A} = \frac{1}{N!} \sum_{R \in S_N} (-1)^R R, \tag{1}$$

where $R \in S_N$ are permutations belonging to a group of all permutations of N objects, S_N. Prove the following properties of $\mathcal{A}$:

   (a) $\mathcal{A}R = (-1)^R \mathcal{A}$, for every $R \in S_N$,

   (b) $\mathcal{A}^2 = \mathcal{A}$ [using (a)].

2. (4 pts.) Use an analysis similar to that carried out in one of our classes to prove the following anti-commutation relation:

$$\{X_p, X_q\} = 0, \tag{2}$$

where X_p is the annihilation operator associated with the single-particle state $|p\rangle$. Use Eq. (2) to show that

$$\{X_p^\dagger, X_q^\dagger\} = 0, \tag{3}$$

where $X_p^\dagger$ is the creation operator associated with the single-particle state $|p\rangle$.

3. (3 pts.) Prove the following relationships:

$$[X_p, N_q] = \delta_{pq} X_p \tag{4}$$

and

$$[X_p^\dagger, N_q] = -\delta_{pq} X_p^\dagger, \tag{5}$$

where X_p, $X_p^\dagger$, and N_p are, respectively, the annihilation, creation, and number operators associated with the single-particle state $|p\rangle$.

4. (3 pts.) Use the anticommutation rules for the creation and annihilation operators to demonstrate that the total number operator,

$$N = \sum_r X_r^\dagger X_r, \tag{6}$$

commutes with the one-body operator

$$Z = \sum_{p,q} \langle p|z|q\rangle X_p^\dagger X_q. \tag{7}$$

5. (3 pts.) Let $|\{pq\}\rangle = X_p^\dagger X_q^\dagger|0\rangle$ and $|\{rst\}\rangle = X_r^\dagger X_s^\dagger X_t^\dagger|0\rangle$ be the configurations belonging to the two- and three-particle Hilbert spaces, $\mathcal{H}_2$ and $\mathcal{H}_3$, respectively. We use the usual notation, where $X_p^\dagger$ is the creation operator associated with the single-particle state $|p\rangle$ and $|0\rangle$ is the vacuum state. Use the anti-commutation relations for the creation and annihilation operators to show that configurations $|\{pq\}\rangle$ and $|\{rst\}\rangle$ are orthogonal.

6. (3pts.) Calculate the following normal products without or with contractions:

   (a) $n[X_p X_q X_r^\dagger X_s^\dagger]$,

   (b) $n[X_p X_q^\dagger X_r X_s^\dagger]$,

   (c) $n[\wick{\c1 X_p \c1 X_q} X_r^\dagger X_s^\dagger]$,

   (d) $n[\wick{\c1 X_p X_q X_r^\dagger \c1 X_s^\dagger}]$,

   (e) $n[\wick{\c1 X_p \c2 X_q \c1 X_r^\dagger \c2 X_s^\dagger}]$,

   (f) $n[\wick{\c1 X_p \c2 X_q \c2 X_r^\dagger \c1 X_s^\dagger}]$.