

Lemma about weights:

$$N_R = w_R / (w_A w_B) = g_A g_B / g_R.$$

Sketch of the proof:

Diagram A consists of q equivalence classes $1, \dots, q$; class k contains m_k equivalent elements ($k=1, \dots, q$).

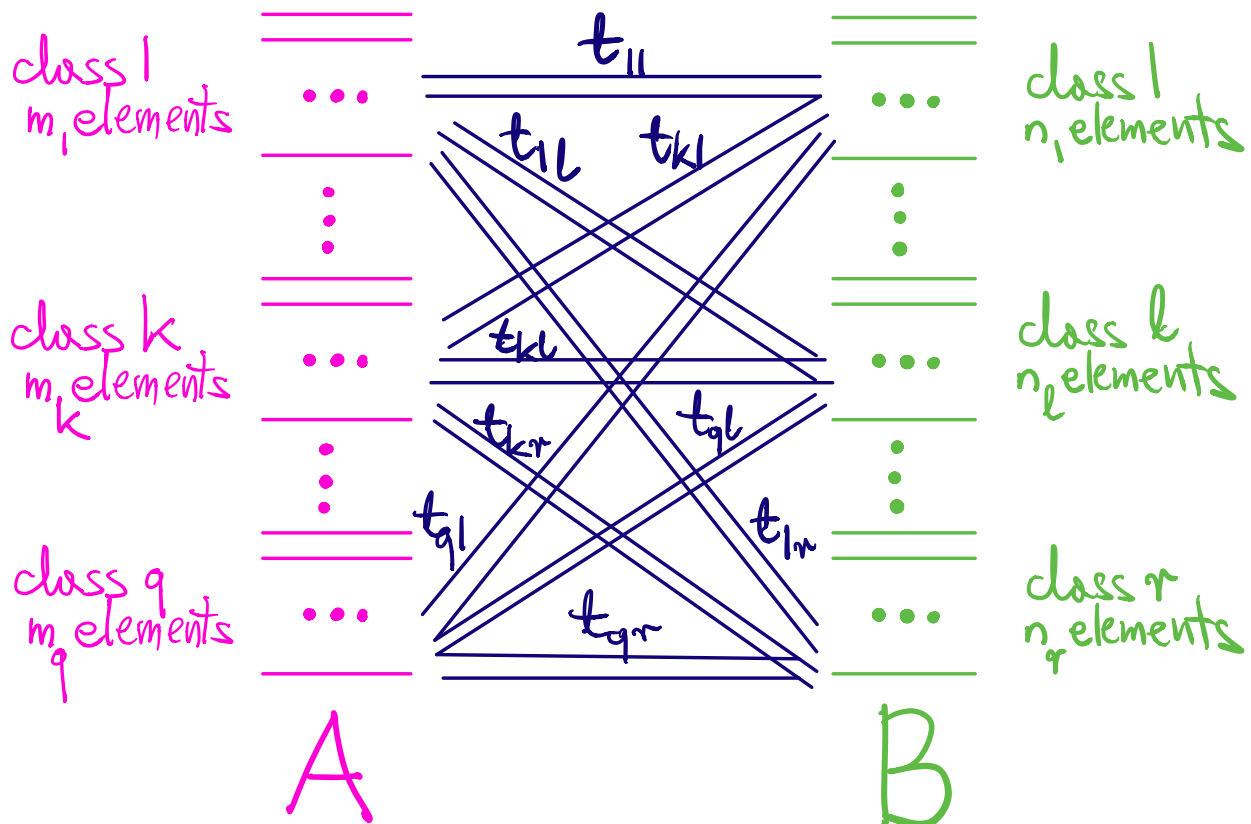
Diagram B consists of r equivalence classes $1, \dots, r$; class l contains n_l equivalent elements ($l=1, \dots, r$).

Define:

$t_{kl} = \#$ of lines connecting class k of A with class l of B.

$t_k^A = \sum_{l=1}^r t_{kl} = \#$ of lines connecting class k of A with all classes of B.

$$t_l^B = \sum_{k=1}^q t_{kl} = \# \text{ of lines connecting class } l \text{ of } B \text{ with all classes of } A.$$



Schematic representation of R

We have:

$$g_A = \prod_{k=1}^q m_k!, \quad g_B = \prod_{l=1}^r n_l!$$

$$g_R = \prod_{k=1}^g (m_k - t_k^A)! \prod_{l=1}^r (n_l - t_l^B)! \\ \times \prod_{k,l} t_{kl}!$$

At the same time,

$$N_R = \left(\prod_{k=1}^g M_k \right) \left(\prod_{l=1}^r N_l \right) \prod_{k,l} t_{kl}!,$$

where M_k is the number of all possible connections for lines from the equivalence class k of A with lines of B , as defined by the t_{kl} numbers with $l=1, \dots, r$, and N_l is the number of all possible connections for lines from the equi-

valence class l of B with lines of A , as defined by the t_{kl} numbers with $k=1, \dots, q$.

We obtain,

$$M_k = \begin{pmatrix} m_k \\ t_{k1} \dots t_{kr} \end{pmatrix},$$

$$N_l = \begin{pmatrix} n_l \\ t_{l1} \dots t_{ql} \end{pmatrix}.$$

Using the above equations for N_R , g_A , g_B , and g_R , we show that

$$\begin{aligned} N_R &= g_A g_B / g_R \\ &= w_R / (w_A w_B). \end{aligned}$$

WICK'S THEOREM IN DIAGRAMMATIC FORM

(rules for resulting d-ms)

Theorem: If

$$K_A = S_A^{-1} \sum_{x_A', x_A''} d_{x_A', x_A''}^{(A)} \hat{O}_{x_A''}$$

...

$$K_Z = S_Z^{-1} \sum_{x_Z', x_Z''} d_{x_Z', x_Z''}^{(Z)} \hat{O}_{x_Z''}$$

are operators corresponding to
diagrams $A, \dots, Z$, respectively,
then

$$K_A \dots K_Z = \sum_R K_R,$$

where the summation over R includes

all nonequivalent diagrams

$$K_R = S_R H_R \sum_{x_R' x_R''} d_{x_R' x_R''}^{(R)} \hat{O}_{x_R''}^{x_R'}$$

with

$$S_R = (-1)^{l_R + h_R}$$

where l_R is the number of loops in R and h_R is the number of internal hole lines in R .

Clearly, the sign determination requires the use of the Goldstone representation. (the case of Hugenholtz diagrammatics will be considered later).

Proof:

Let us assume that

$$K_A = w_A \sum_{X_A''} d_{X_A''}^{(A)} \hat{O}_{X_A''},$$

...

$$K_Z = w_Z \sum_{X_Z''} d_{X_Z''}^{(Z)} \hat{O}_{X_Z''},$$

where

$$\begin{aligned} \hat{O}_{X_A''} &= N \left[X_{p_1^{(A)}}^+ \dots X_{p_\alpha^{(A)}}^+ X_{q_2^{(A)}} \dots X_{q_1^{(A)}} \right] \\ &= N \left[X_{p_1^{(A)}}^+ X_{q_1^{(A)}} \dots X_{p_\alpha^{(A)}}^+ X_{q_\alpha^{(A)}} \right], \\ &\dots \end{aligned}$$

$$\begin{aligned}\hat{O}_{X''} &= N[X_{p_1}^{+ (2)} \dots X_{p_m}^{+ (2)} X_{q_1}^{(2)} \dots X_{q_l}^{(2)}] \\ &= N[X_{p_1}^{+ (2)} X_{q_1}^{(2)} \dots X_{p_m}^{+ (2)} X_{q_l}^{(2)}],\end{aligned}$$

(normal-ordered operators)

We can write,

$$\begin{aligned}K_A \dots K_Z &= W_A \dots W_Z \\ &\times \sum_{X''_A, \dots, X''_Z} d_{X''_A}^{(A)} \dots d_{X''_Z}^{(Z)}\end{aligned}$$

$$\times N[X_{p_1}^{+ (A)} X_{q_1}^{(A)} \dots X_{p_\alpha}^{+ (A)} X_{q_\alpha}^{(A)}]$$

$$\times \dots N[X_{p_1}^{+ (Z)} X_{q_1}^{(Z)} \dots X_{p_m}^{+ (Z)} X_{q_l}^{(Z)}]$$

$$= \sum_R' K_R' = \sum_R N_R K_R' = \sum_R K_R$$

summation over
all terms of WT
(including
repetitions)

summation over
nonequivalent
terms, with
 N_R representing
the # of times
one obtains K_R



where

$$K_R = N_R K_R' = N_R W_A \cdots W_Z$$

W_R
(from
our
lemma)

$$\times \sum_{x_A'', \dots, x_Z''} d_{x_A''}^{(A)} \cdots d_{x_Z''}^{(Z)}$$

$$\times N \left[\overset{+}{X}_{p_1}^{(A)} \overset{+}{X}_{q_1}^{(A)} \cdots \overset{+}{X}_{p_\alpha}^{(A)} \overset{+}{X}_{q_\alpha}^{(A)} \cdots \overset{+}{X}_{p_1}^{(Z)} \overset{+}{X}_{q_1}^{(Z)} \cdots \overset{+}{X}_{p_\alpha}^{(Z)} \overset{+}{X}_{q_\alpha}^{(Z)} \right]$$

from zero $\rightarrow$ some contractions of X^+ and X
to f.c. operators

We can rewrite K_R as follows:

$$K_R = W_R \sum_{x'_R, x''_R} d_{x'_R x''_R}^{(R)}$$

indices of
internal lines
produced by
contractions

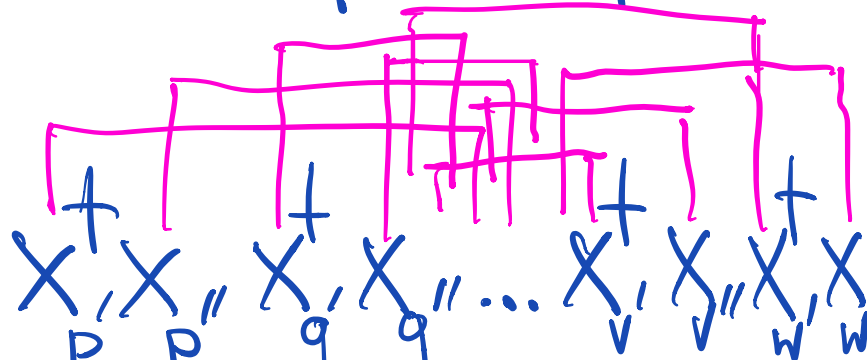
indices of
uncontracted
external lines

$$\times N[(\text{loop})_1 \dots (\text{loop})_L (\text{open path})_1 \dots (\text{open path})_M]$$

where

Kronecker deltas from
contractions

$$d_{x'_R x''_R}^{(R)} \Leftarrow d_{x''_A}^{(A)} \dots d_{x''_Z}^{(Z)} \delta_{x'_R}$$

$$\text{loop} = \overset{+}{X'_P} \overset{+}{X''_P} \overset{+}{X'_Q} \overset{+}{X''_Q} \dots \overset{+}{X'_V} \overset{+}{X''_V} \overset{+}{X'_W} \overset{+}{X''_W}$$


and
 open path = $X_{p'}^+ X_{p''} X_{q'}^+ X_{q''} \dots X_{v'}^+ X_{v''} X_{w'}^+ X_{w''}$

The diagram shows a sequence of nodes: $X_{p'}^+$, $X_{p''}$, $X_{q'}^+$, $X_{q''}$, ..., $X_{v'}^+$, $X_{v''}$, $X_{w'}^+$, $X_{w''}$. Pink arcs connect $X_{p'}^+$ to $X_{q'}^+$, $X_{p''}$ to $X_{q''}$, $X_{q'}^+$ to $X_{v'}^+$, $X_{q''}$ to $X_{v''}$, and $X_{v'}^+$ to $X_{w'}^+$. Green arrows point upwards to the first node $X_{p'}^+$ and the last node $X_{w''}$.

Here we used the following
 observations:

$X_{p'}^+ X_{p''} \dots X_{q'}^+ X_{q''} \dots X_{v'}^+ X_{v''} \dots X_{w'}^+ X_{w''}$

lead to open paths

The diagram shows a sequence of node pairs: $X_{p'}^+ X_{p''}$, ..., $X_{q'}^+ X_{q''}$, ..., $X_{v'}^+ X_{v''}$, ..., $X_{w'}^+ X_{w''}$. Pink arcs connect the first node of one pair to the first node of the next pair, and the second node of one pair to the second node of the next pair. Wavy pink lines are under the first and last pairs.

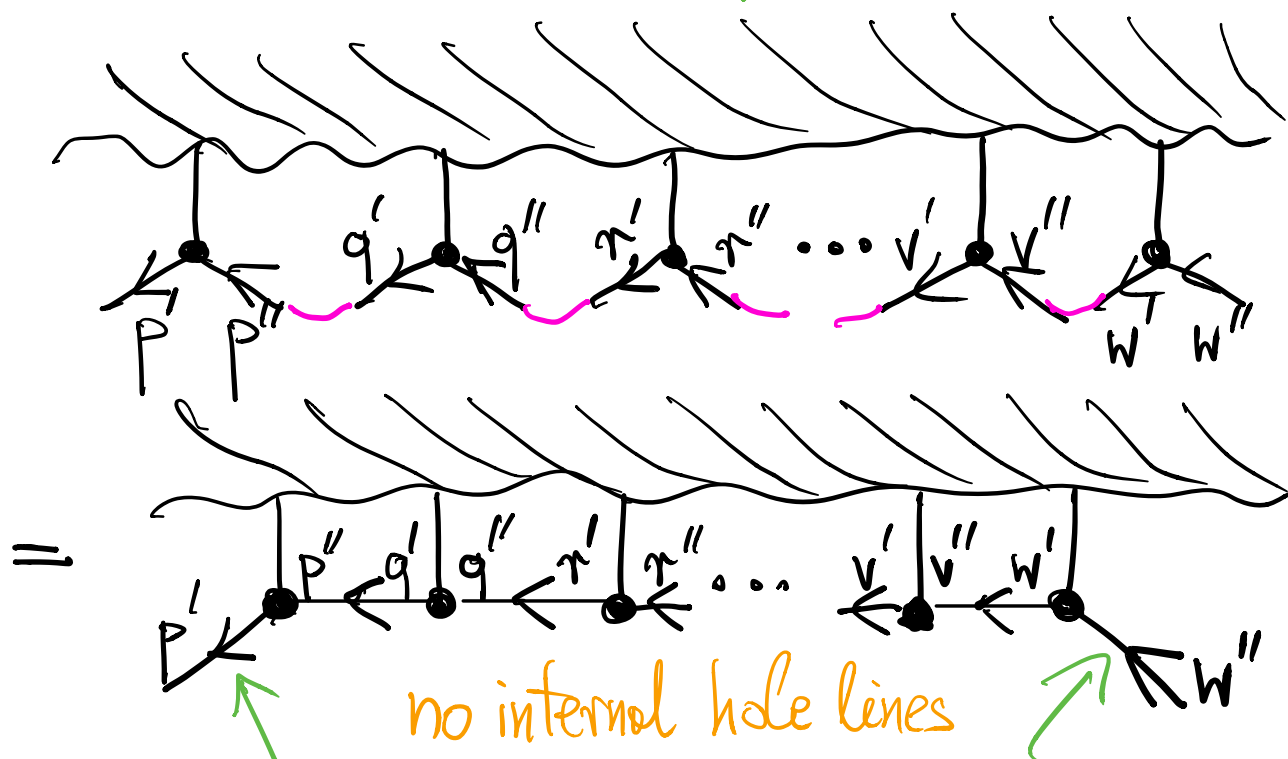
$X_{p'}^+ X_{p''} \dots X_{q'}^+ X_{q''} \dots X_{v'}^+ X_{v''} \dots X_{w'}^+ X_{w''}$

lead to loops

The diagram shows a sequence of node pairs: $X_{p'}^+ X_{p''}$, ..., $X_{q'}^+ X_{q''}$, ..., $X_{v'}^+ X_{v''}$, ..., $X_{w'}^+ X_{w''}$. Pink arcs connect the first node of one pair to the first node of the next pair, and the second node of one pair to the second node of the next pair. A long pink arc connects the first node of the first pair to the first node of the last pair, and another long pink arc connects the second node of the first pair to the second node of the last pair.

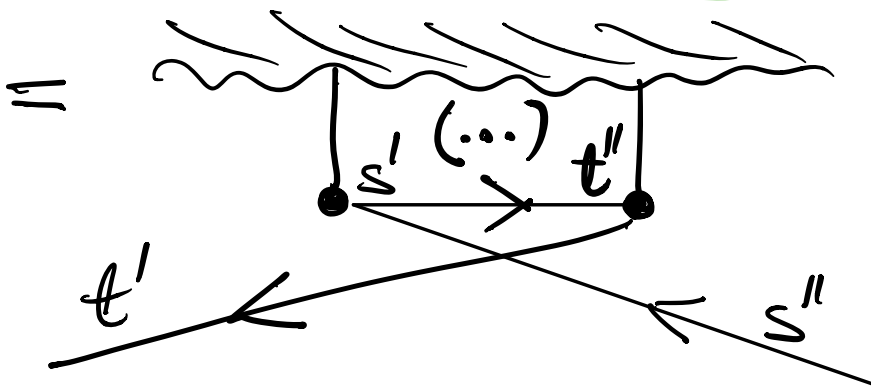
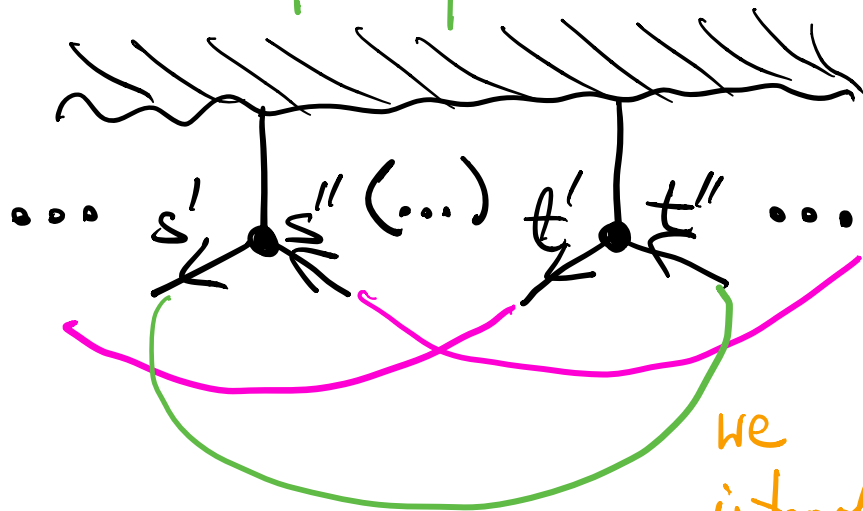
Let us now "unscramble" open paths and loops. To do this, we consider 5 cases and the role of the number of internal hole lines in open paths and loops. We will start with open paths.

(I) [canonical open path]



$$\begin{aligned}
 & N[\dots X_{p'}^+ X_{p''}^+ X_{q'}^+ X_{q''}^+ \dots X_{v'}^+ X_{v''}^+ X_{w'}^+ X_{w''}^+ \dots] \\
 &= + X_{p''}^+ X_{q'}^+ X_{q''}^+ X_{r'}^+ \dots X_{v''}^+ X_{w'}^+ \\
 &\quad \times N[\dots X_{p'}^+ X_{w''}^+ \dots]
 \end{aligned}$$

(II)

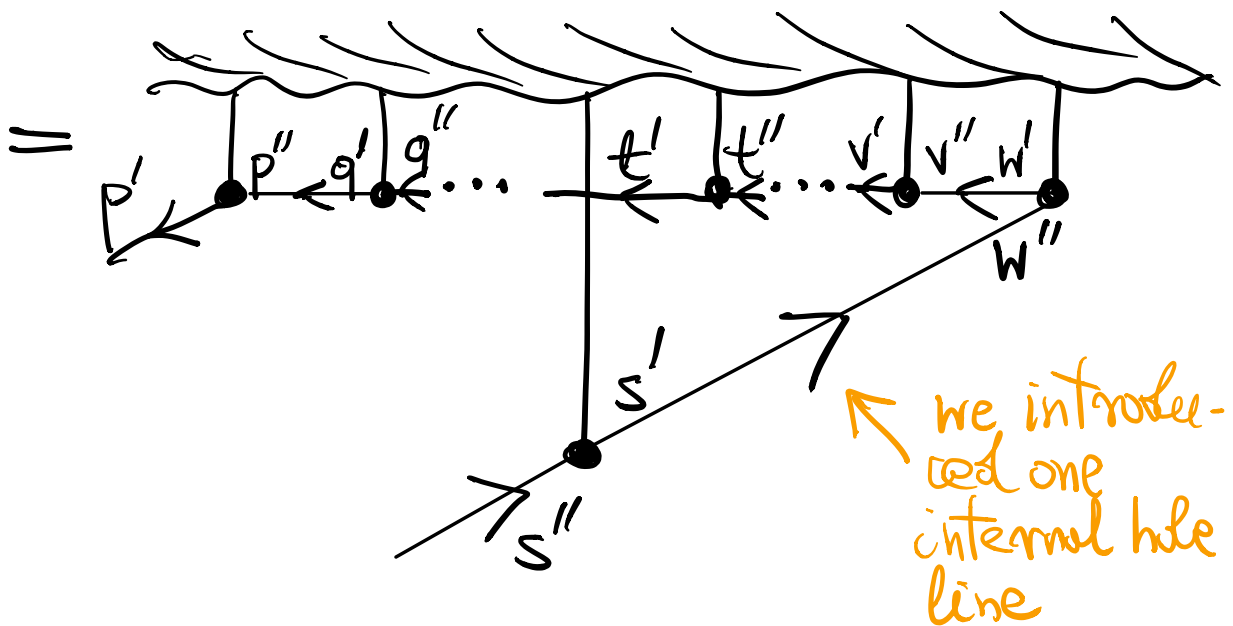
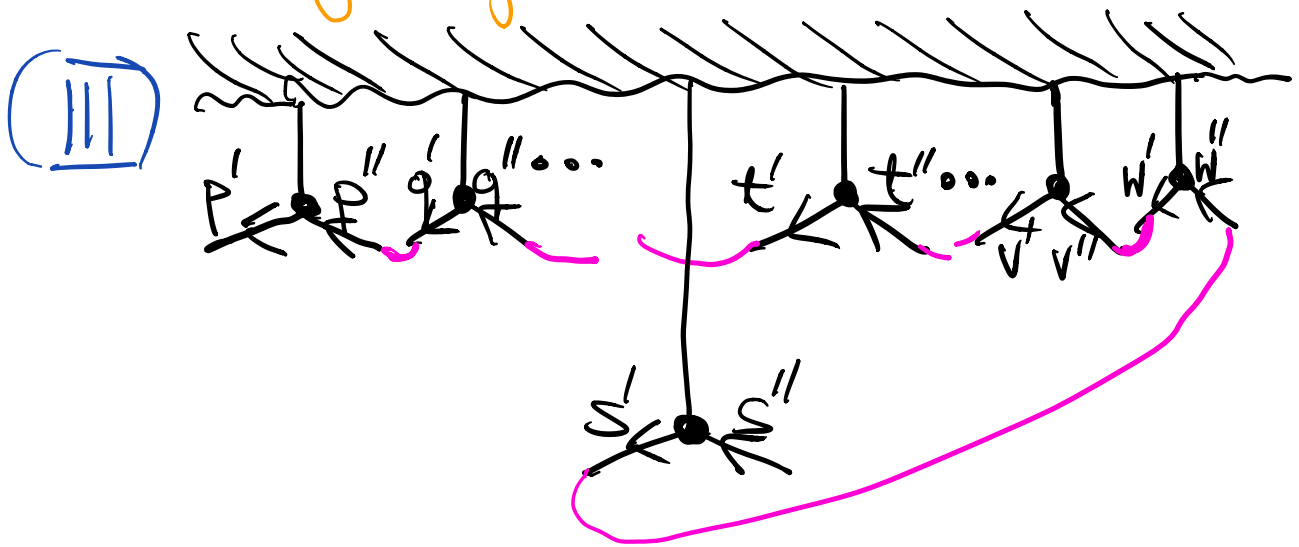


we introduced
one
internal
hole line
($s't''$)

$$N[\dots X_{s'}^+ X_{s''} \dots X_{t'}^+ X_{t''} \dots]$$

$$= - X_{s'}^+ X_{t''} N[\dots X_{t'}^+ X_{s''} \dots]$$

↑ sign change ← internal hole line



$$N[\dots X_{p'}^{\dagger} X_{p''} X_{q'}^{\dagger} X_{q''} \dots X_{s'}^{\dagger} X_{s''} X_{t'}^{\dagger} X_{t''} \dots X_{v'}^{\dagger} X_{v''} X_{w'}^{\dagger} X_{w''} \dots]$$

Diagram showing a sequence of operators in a normal ordering bracket. Green arrows point to $X_{p'}$ and $X_{s'}$. A long magenta bracket connects $X_{s'}$ to $X_{w''}$, with a dot at its left end. Shorter magenta brackets connect $X_{p''}$ to $X_{q'}$, $X_{q''}$ to $X_{s'}$, $X_{s''}$ to $X_{t'}$, $X_{t''}$ to $X_{v'}$, $X_{v''}$ to $X_{w'}$, and $X_{w''}$ to $X_{w''}$.

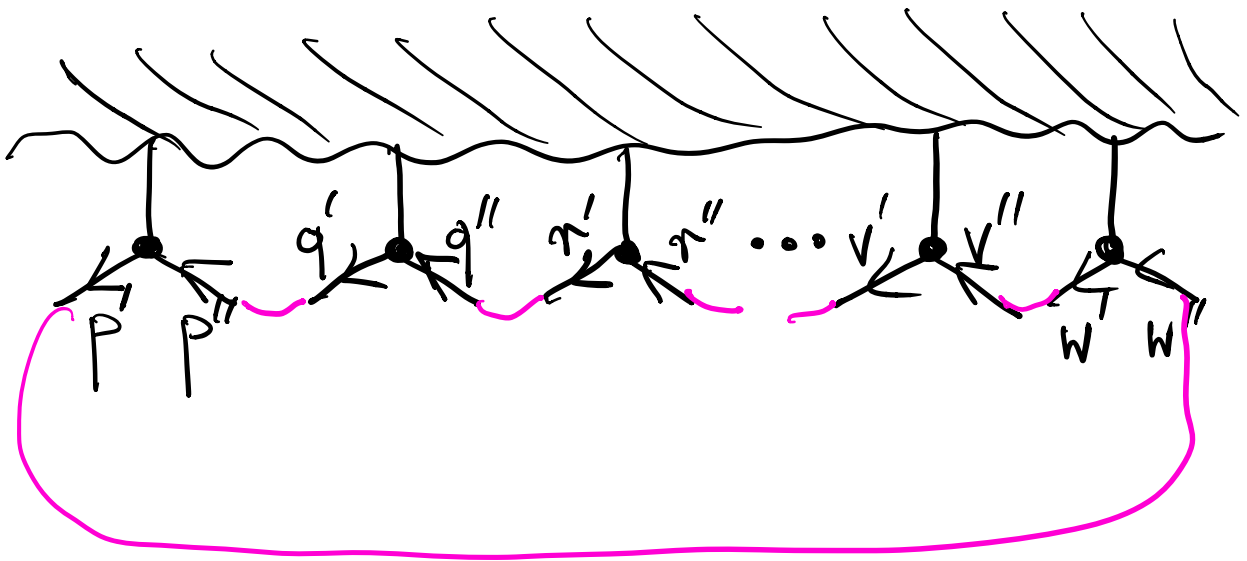
$$= - X_{p''} X_{q'}^{\dagger} \dots X_{v''} X_{w'}^{\dagger} \underbrace{X_{s'}^{\dagger} X_{w''}}_{\text{internal hole line}} \times N[\dots X_{p'}^{\dagger} X_{s''} \dots]$$

Diagram showing the result of the contraction. The first part is a product of operators with magenta brackets connecting $X_{p''}$ to $X_{q'}$, $X_{v''}$ to $X_{w'}$, and $X_{s'}^{\dagger}$ to $X_{w''}$. The last bracket is labeled "internal hole line" with a wavy orange line underneath. The second part is a normal ordering bracket containing $X_{p'}^{\dagger}$ and $X_{s''}$, with green arrows pointing to each.

Loops:

(IV)

[canonical loop] (analogy of I)

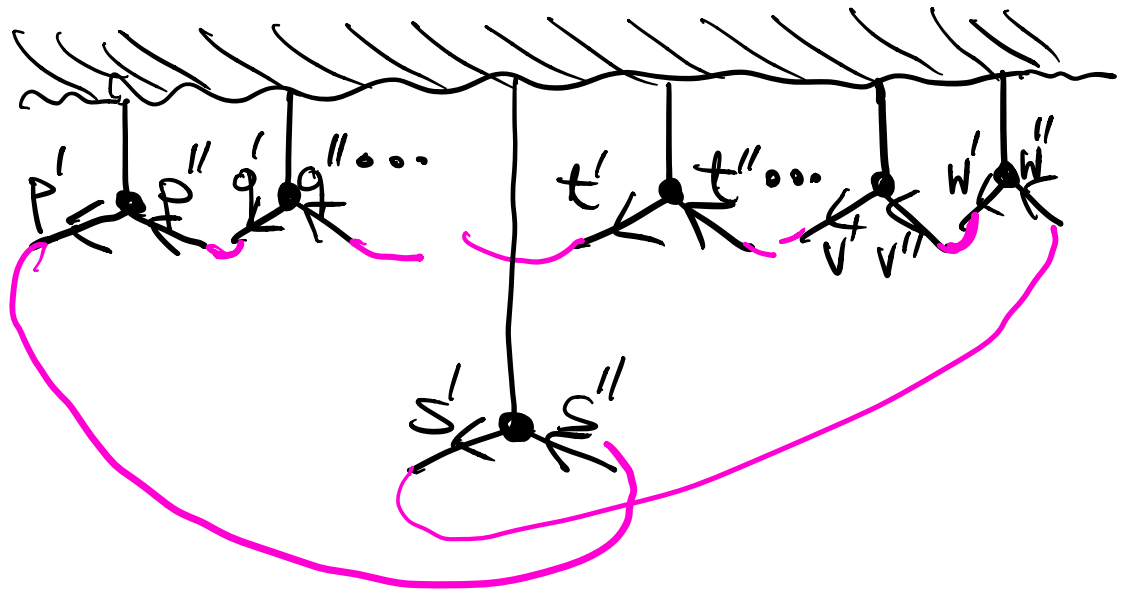


exactly
one internal
hole line
(minimum for a loop)

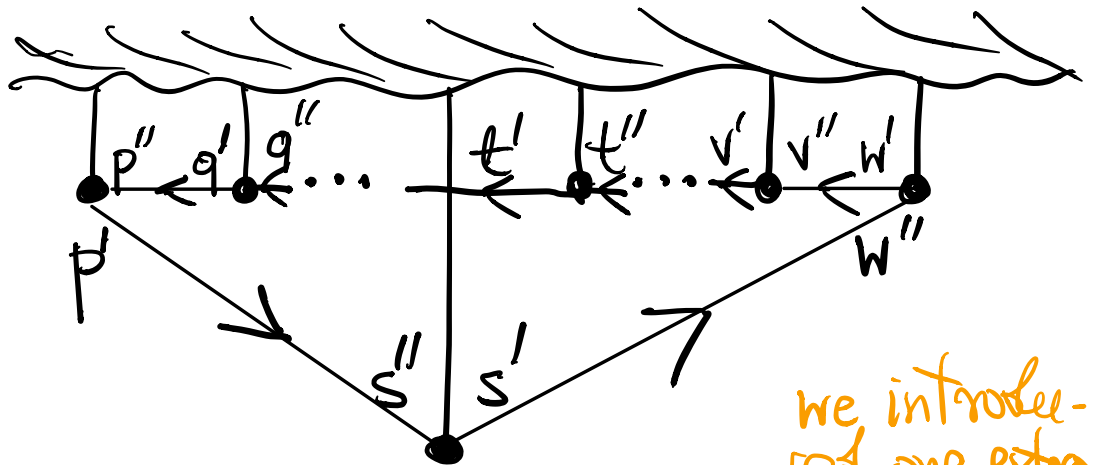
$$\begin{aligned}
& N \left[\overbrace{X_p^+ X_p'' X_q^+ X_q''} \dots \overbrace{X_v^+ X_v'' X_w^+ X_w''} \dots \right] \\
&= + \overbrace{X_p'' X_q^+ X_q'' X_q^+} \dots \overbrace{X_v'' X_w^+} \\
&\quad \times \underbrace{X_p^+ X_w''}_{\text{hole line}} N[\dots]
\end{aligned}$$

Introduction of more internal hole lines can be done using case (I) ($\Delta h = 1$, sign will change) or an analogue of (III), called (V).

(V)



||



we introduced one extra internal hole line to canonical loop

$$N[\dots X_{p'}^+ X_{p''}^- X_{q'}^+ X_{q''}^- \dots X_{s'}^+ X_{s''}^- X_{t'}^+ X_{t''}^- \dots X_{v'}^+ X_{v''}^- X_{w'}^+ X_{w''}^- \dots]$$

$$= - X_{p''}^- X_{q'}^+ \dots X_{v''}^- X_{w'}^+ X_{s'}^+ X_{w''}^- \times X_{p'}^+ X_{s''}^- N[\dots]$$

internal hole line

internal hole line

Summary:

(I) $h=0, +$

(II) $\Delta h=1, -$

(III) $\Delta h=1, -$

(IV) $h=1, +$

[(III)] $\Delta h=1, -$

(V) $\Delta h=1, -$

open paths

loops

$$\begin{aligned}
 & N[(\text{loop})_1 \dots (\text{loop})_{l_R} (\text{open path})_1 \dots (\text{open path})_{m_R}] \\
 &= \delta_{\chi_R'} N \left[\underbrace{X_{p_1}^{(R)} X_{q_1}^{(R)} \dots X_{p_{m_R}}^{(R)} X_{q_{m_R}}^{(R)}}_{\substack{\text{external lines} \\ \text{of } (\text{open path})_1}} \right] \\
 &\quad \times \delta_{\chi_R}, \text{ where}
 \end{aligned}$$

$$\delta_R = \prod_{\lambda=1}^{l_R} (-1)^{h_{\lambda}+1} \prod_{\mu=1}^{m_R} (-1)^{h_{\mu}},$$

with

$$h_{\gamma} = \# \text{ of internal hole lines} \\ \text{in } (\text{loop})_{\gamma}$$

and

$$h_{\mu} = \# \text{ of internal hole lines} \\ \text{in } (\text{open path})_{\mu}.$$

We obtain

$$S_R = (-1)^{l_R + h_R}$$

$$h_R = \sum_{\gamma=1}^{l_R} h_{\gamma} + \sum_{\mu=1}^{m_R} h_{\mu}$$

$$= \# \text{ of all internal} \\ \text{hole lines in } R.$$

Thus,

$$K_A \dots K_Z = \sum_R K_R, \text{ where}$$

$$K_R = S_R W_R \sum_{x_R' x_R''} d_{x_R' x_R''}^{(R)}$$

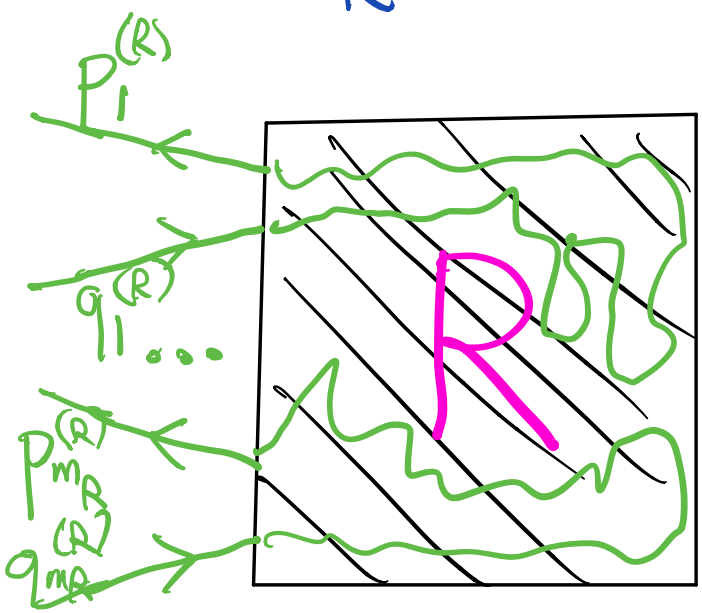
$$\times N \left[\prod_{r=1}^{m_R} X_{p_r}^{(R)} X_{q_r}^{(R)} \right]$$

product of scalars associated with $A, \dots, Z$

and

$$S_R = (-1)^{l_R + h_R}$$

x_R'' consists of $p_r^{(R)}, q_r^{(R)}, r=1, \dots, m_R$.



Comments:

How about standard operator ordering?

$$K_C = w_C \sum_{\chi_C''} d_{\chi_C''}^{(C)} \hat{O}_{\chi_C''},$$

where

$$\hat{O}_{\chi_C''} = X_{p_1}^{+ (C)} \dots X_{p_g}^{+ (C)} X_{q_g (C)} \dots X_{q_1 (C)}$$

We now allow contractions within each operator K_C written in a standard form. These contractions are

$$\overbrace{X_p^+ X_q} = \chi(p) \delta_{pq}$$

meaning that the resulting internal lines are HOLE lines.

The theorem and the rules remain the same as in the case of normal-ordered operators $K_A, \dots, K_Z$, but we allow contractions originating from the same vertex and label the corresponding internal lines with hole indices $(i, j, k, \dots)$.

Situations of common interest:

$$K_A \dots K_Z |\Phi\rangle ,$$

$$\langle \Phi | K_A \dots K_Z |\Phi\rangle .$$

$$\langle \Phi | K_A \dots K_Z | \Phi \rangle = \sum_{\substack{R \\ \text{f.c.}}} \langle \Phi | K_R | \Phi \rangle$$

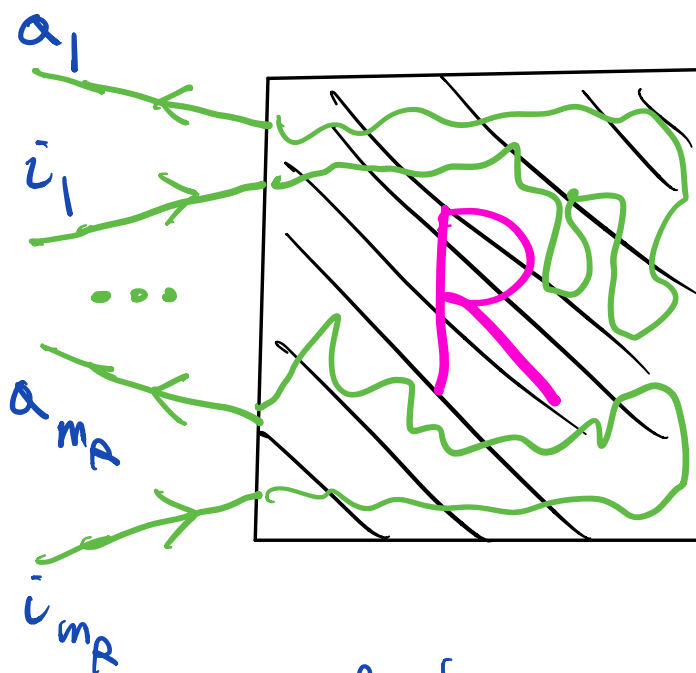
[NO EXTERNAL
LINES]

$$K_R = S_R W_R \sum_{\chi_R'} d_{\chi_R'}^{(R)} =$$

$$= (-1)^{L_R + h_R} W_R \sum_{\chi_R'} d_{\chi_R'}^{(R)}.$$

$$K_A \dots K_Z | \Phi \rangle = \sum_{R, \text{a.f.c.}} K_R | \Phi \rangle$$

[ALL EXTERNAL
LINES MUST
EXTEND TO THE
LEFT]



$$K_R |\Phi\rangle = (-1)^{l_R + h_R} w_R$$

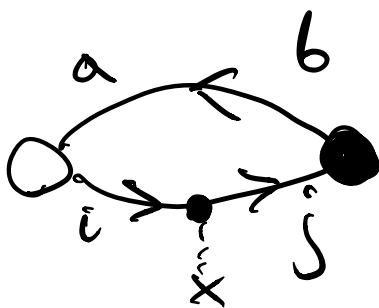
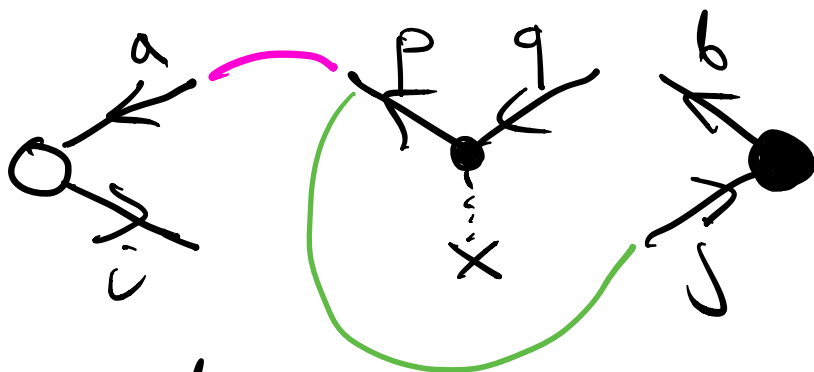
$$\times \sum_{\mathcal{X}_R} \sum_{\substack{a_1 \dots a_{m_R} \\ i_1 \dots i_{m_R}}} d_{\mathcal{X}_R, a_1 \dots a_{m_R}, i_1 \dots i_{m_R}}^{(R)} \\ \times \left| \Phi_{\substack{a_1 \dots a_{m_R} \\ i_1 \dots i_{m_R}}} \right\rangle$$

$$\left(\prod_{r=1}^{m_R} X_{a_r}^\dagger X_{i_r} |\Phi\rangle \right)$$

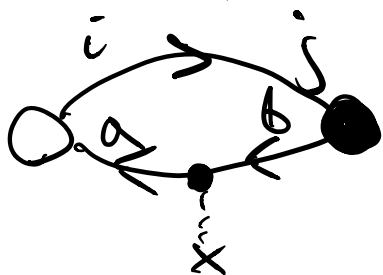
Examples:

$$\begin{aligned} & \langle \Phi_i^a | Z_N | \Phi_j^b \rangle \\ &= \langle \Phi | (E_i^a)^\dagger Z_N E_j^b | \Phi \rangle \\ & (E_i^a = N[X_a^\dagger X_i]) \end{aligned}$$

pq-free
a, i, b, j -
fixed
mm



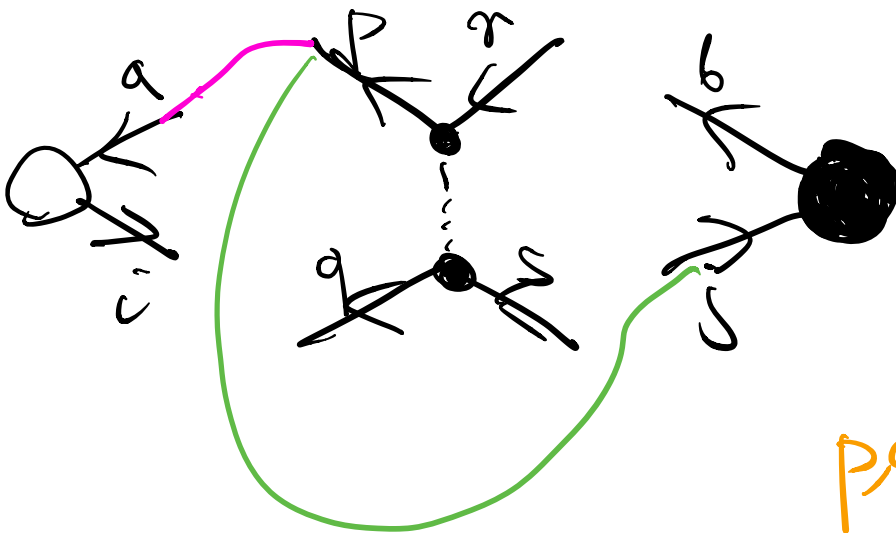
$$w_R = 1, S_R = (-1)^{1+2} = -1$$



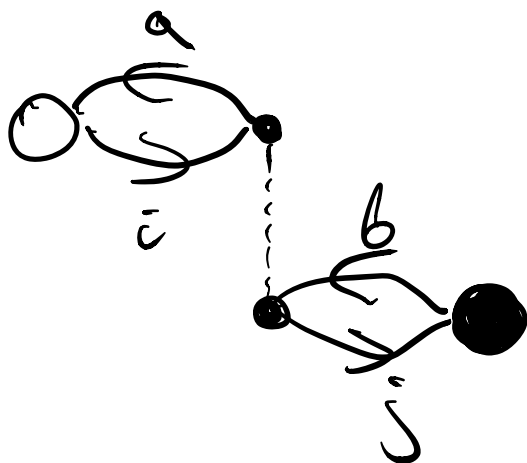
$$w_R = 1, S_R = (-1)^{1+1} = +1$$

$$\langle \Phi_i^a | Z_N | \Phi_j^b \rangle = - \langle j | \hat{z} | i \rangle \delta_{ab} \\ + \langle a | \hat{z} | b \rangle \delta_{ij}$$

$$\langle \Phi_i^a | V_N | \Phi_j^b \rangle \\ = \langle \Phi_i^a | (E_i^a)^\dagger V_N E_j^b | \Phi \rangle$$

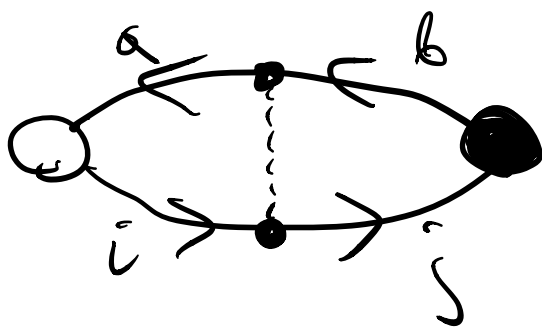


p, q, r, s - free
 a, i, b, j - fixed
 mn



$$w_R = 1$$

$$S_R = (-1)^{2+2} = +1$$



$$w_R = 1$$

$$S_R = (-1)^{1+2} = -1$$

$$\langle \Phi_i^a | V_N | \Phi_j^b \rangle = \langle a_j | \hat{v} | i b \rangle$$

$$- \langle a_j \hat{v} | b i \rangle$$

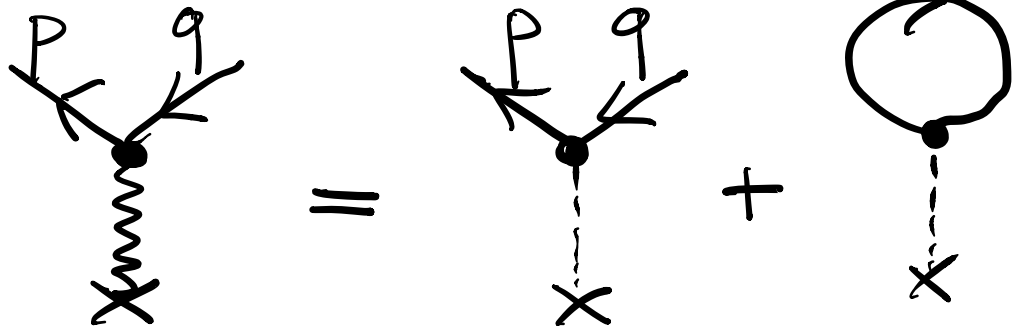
$$= \langle a_j | \hat{v} | i b \rangle_A$$

Relationships between
 Z and Z_N and V and V_N
 diagrammatically (to practice
 diagrams originating from
 standard operator ordering).

Z :

p, q free

free

$$w_R = 1$$

$$S_R = (-1)^{0+0} = +1$$

$$(h_R = 0, l_R = 0)$$

$$w_R = 1$$

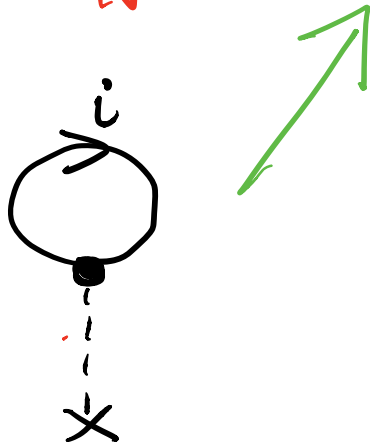
$$S_R = (-1)^{1+1} = +1$$

$$(h_R = 1, l_R = 1)$$

$$Z = \sum_{p,q} \langle p | \hat{Z} | q \rangle N[X_p^\dagger X_q]$$

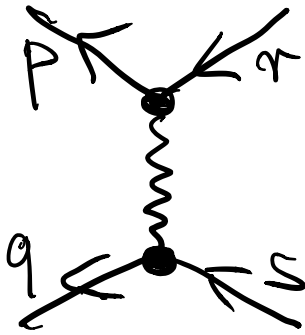
$$+ \sum_i \langle i | \hat{Z} | i \rangle$$

$$= Z_N + \langle \Phi | Z | \Phi \rangle$$

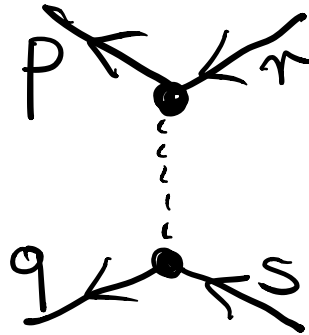


V:

p, q, r, s -free

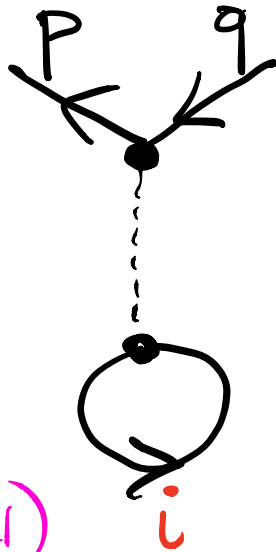


=



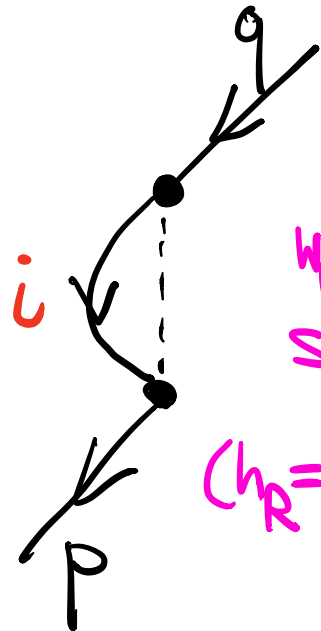
$w_R = \frac{1}{2}$
 $s_R = +1$
 $(h_R = 0, l_R = 0)$

+



$w_R = 1$
 $s_R = +1$
 $(h_R = 1, l_R = 1)$

+



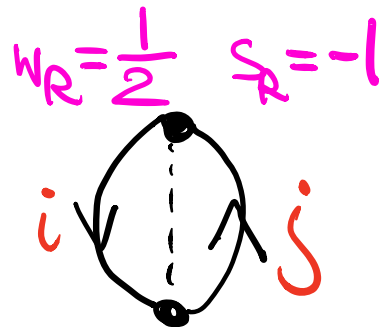
$w_R = 1$
 $s_R = -1$
 $(h_R = 1, l_R = 0)$

+



$w_R = \frac{1}{2}$
 $s_R = +1$ $(h_R = 2, l_R = 2)$

+



$w_R = \frac{1}{2}$ $s_R = -1$
 $h_R = 2, l_R = 1$

$$V = \frac{1}{2} \sum_{pq,rs} \langle pq|\hat{v}|rs \rangle N[X_p^\dagger X_r X_q^\dagger X_s]$$

$$+ \sum_{pq} \sum_i \langle p\bar{i}|\hat{v}|q\bar{i} \rangle N[X_p^\dagger X_q]$$

$$- \sum_{pq} \sum_i \langle p\bar{i}|\hat{v}|\bar{i}q \rangle N[X_p^\dagger X_q]$$

$$+ \frac{1}{2} \sum_{\bar{i}\bar{j}} \langle \bar{i}\bar{j}|\hat{v}|\bar{i}\bar{j} \rangle$$

$$- \frac{1}{2} \sum_{\bar{i}\bar{j}} \langle \bar{i}\bar{j}|\hat{v}|\bar{j}\bar{i} \rangle$$

$$V = V_N + \sum_{pq} \sum_i \langle p\bar{i}|\hat{v}|q\bar{i} \rangle_A N[X_p^\dagger X_q]$$

$$+ \frac{1}{2} \sum_{\bar{i}\bar{j}} \langle \bar{i}\bar{j}|\hat{v}|\bar{i}\bar{j} \rangle_A$$

$$= V_N + G_N + \langle \Phi | V | \Phi \rangle$$

$$G_N = \sum_{pq} \langle p | g | q \rangle N[x_p^\dagger x_q]$$

$$\langle p | g | q \rangle = \sum_c \langle p \bar{c} | v | q \bar{c} \rangle_A$$

$$\langle \Phi | V | \Phi \rangle = \frac{1}{2} \sum_{\bar{c} \bar{d}} \langle \bar{c} | v | \bar{d} \rangle_A$$

