Lemma about weights:

$$N_R = w_R/(w_A w_B) = g_A g_B / g_R .$$

Sketch of the proof:

Diagram A consists of q equivalence classes $1,\ldots,q$; class k contains m_k equivalent elements $(k=1,\ldots,q)$.

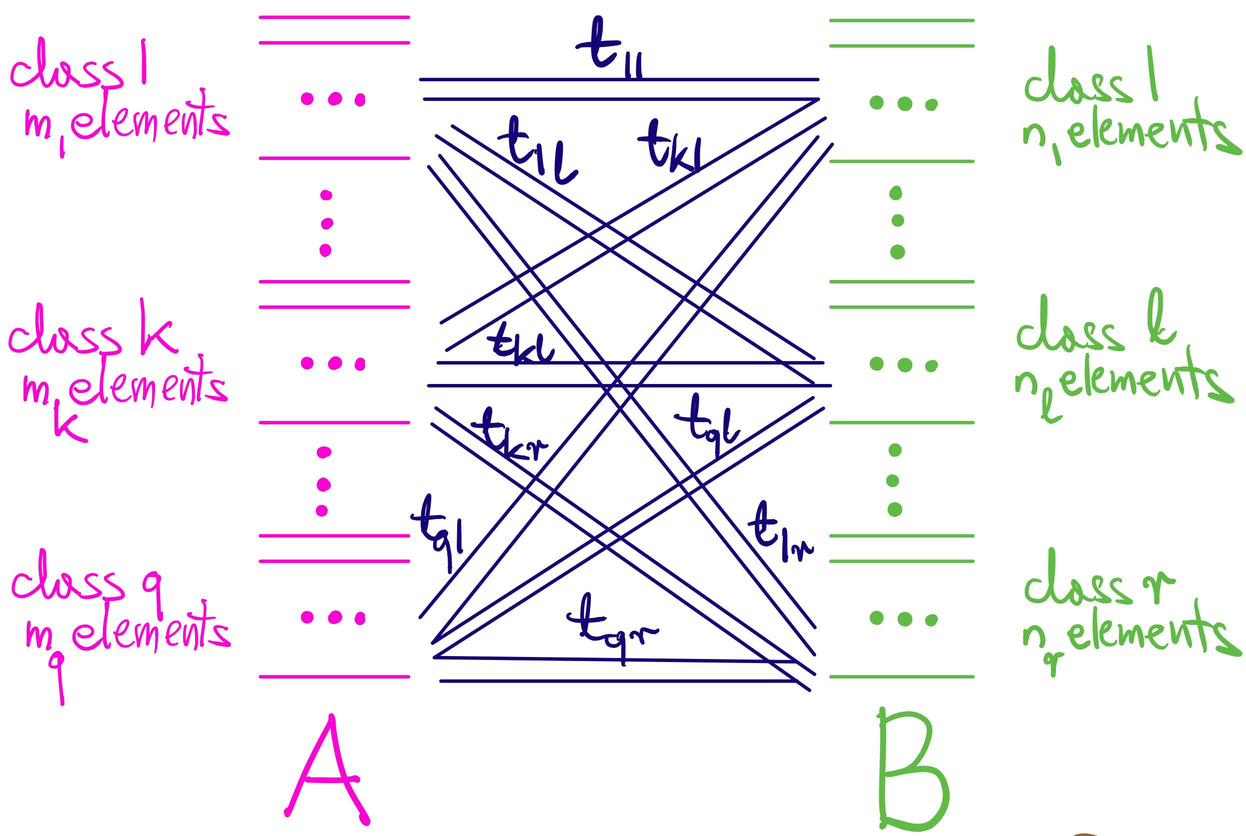
Diagram B consists of r equivalence classes $1,\ldots,r$; class l contains n_l equivalent elements $(l=1,\ldots,r)$.

Define:

t_{kl} = # of lines connecting class k of A with class l of B.

$t_k^A = \sum_{l=1}^{r} t_{kl}$ = # of lines connecting class k of A with all classes of B.

$t_l^B = \sum_{k=1}^{q} t_{kl}$ = # of lines connecting class l of B with all classes of A.



Schematic representation of R

We have:

$$g_A = \prod_{k=1}^{q} m_k! \,, \quad g_B = \prod_{l=1}^{r} n_l! \,,$$

$$g_R = \prod_{k=1}^{q} (m_k - t_k^A)! \prod_{l=1}^{r} (n_l - t_l^B)!$$
$$\times \prod_{k,l} t_{kl}!$$

At the same time,

$$N_R = \left(\prod_{k=1}^{q} M_k\right)\left(\prod_{l=1}^{r} N_l\right) \prod_{k,l} t_{kl}!,$$

where M_k is the number of all possible connections for lines from the equivalence class k of A with lines of B, as defined by the t_{kl} numbers with $l=1,\dots,r$, and N_l is the number of all possible connections for lines from the equivalence class l of B with lines of A, as defined by the t_{kl} numbers with $k=1,\ldots,q$.

We obtain,

$$M_k = \begin{pmatrix} & m_k & \\ t_{k1} & \cdots & t_{kr} \end{pmatrix},$$

$$N_l = \begin{pmatrix} & n_l & \\ t_{1l} & \cdots & t_{ql} \end{pmatrix}.$$

Using the above equations for N_R, g_A, g_B, and g_R, we show that

$$N_R = g_A g_B / g_R$$
$$= w_R / (w_A w_B).$$

WICK's THEOREM IN DIAGRAMMATIC FORM

(rules for resulting d-ms)

Theorem: If

$$K_A = s_A w_A \sum_{x_A', x_A''} d^{(A)}_{x_A' x_A''} \hat{O}_{x_A', x_A''}$$

$$\dots$$

$$K_Z = s_Z w_Z \sum_{x_Z', x_Z''} d^{(Z)}_{x_Z' x_Z''} \hat{O}_{x_Z', x_Z''}$$

are operators corresponding to diagrams $A, \dots, Z$, respectively, then

$$K_A \cdots K_Z = \sum_R K_R ,$$

where the summation over R includes

all nonequivalent diagrams

$$K_R = s_R H_R \sum_{x_R', x_R''} d^{(R)}_{x_R' x_R''} \hat{O}_{x_R''}$$

with

$$s_R = (-1)^{l_R + h_R},$$

where l_R is the number of loops in R and h_R is the number of internal hole lines in R. Clearly, the sign determination requires the use of the Goldstone representation (the case of Hugenholtz diagrammatics will be considered later).

Proof:

Let us assume that

$$K_A = w_A \sum_{\chi_A''} d^{(A)}_{\chi_A''} \hat{O}_{\chi_A''},$$

$$\dots$$

$$K_Z = w_Z \sum_{\chi_Z''} d^{(Z)}_{\chi_Z''} \hat{O}_{\chi_Z''},$$

where

$$\hat{O}_{\chi_A''} = N\left[X^{\dagger}_{p_1^{(A)}} \dots X^{\dagger}_{p_\alpha^{(A)}} X_{q_\alpha^{(A)}} \dots X_{q_1^{(A)}}\right]$$

$$= N\left[X^{\dagger}_{p_1^{(A)}} X_{q_1^{(A)}} \dots X^{\dagger}_{p_\alpha^{(A)}} X_{q_\alpha^{(A)}}\right],$$

$$\dots$$

$$\hat{O}_{x''_Z} = N[X^\dagger_{p_1}(Z) \ldots X^\dagger_{p_\ell}(Z) X_{q_\ell}(Z) \ldots X_{q_1}(Z)]$$
$$= N[X^\dagger_{p_1}(Z) X_{q_1}(Z) \ldots X^\dagger_{p_\ell}(Z) X_{q_\ell}(Z)],$$

(normal-ordered operators)

We can write,

$$K_A \cdots K_Z = w_A \cdots w_Z$$
$$\times \sum_{x''_A, \ldots, x''_Z} d^{(A)}_{x''_A} \cdots d^{(Z)}_{x''_Z}$$
$$\times\ N[X^\dagger_{p_1}(A) X_{q_1}(A) \ldots X^\dagger_{p_\alpha}(A) X_{q_\alpha}(A)]$$
$$\times \ldots N[X^\dagger_{p_1}(Z) X_{q_1}(Z) \ldots X^\dagger_{p_\ell}(Z) X_{q_\ell}(Z)]$$

$$= \sum_R{}' K_R' = \sum_R N_R K_R' = \sum_R K_R$$

summation over all terms of WT (including repetitions)

summation over nonequivalent terms, with N_R representing the # of times one obtains K_R'

where

$$K_R = N_R K_R' = \underbrace{N_R w_A \cdots w_Z}_{= w_R}$$

w_R (from our lemma)

$$\times \sum_{x_A'', \ldots, x_Z''} d^{(A)}_{x_A''} \cdots d^{(Z)}_{x_Z''}$$

$$\times N\left[X^\dagger_{p_1^{(A)}} X_{q_1^{(A)}} \cdots X^\dagger_{p_\alpha^{(A)}} X_{q_\alpha^{(A)}} \cdots X^\dagger_{p_1^{(Z)}} X_{q_1^{(Z)}} \cdots X^\dagger_{p_\xi^{(Z)}} X_{q_\xi^{(Z)}}\right]$$

from zero to f.c. $\longrightarrow$ some contractions of $X^\dagger$ and X operators

We can rewrite K_R as follows:

$$K_R = w_R \sum_{x_R', x_R''} d^{(R)}_{x_R', x_R''}$$

indices of internal lines produced by contractions

indices of uncontracted external lines

$$\times N\left[(\text{loop})_1 \ldots (\text{loop})_{\ell_R} (\text{open path})_1 \ldots (\text{open path})_{m_R}\right]$$

where

$$d^{(R)}_{x_R', x_R''} \Leftarrow d^{(A)}_{x_A''} \ldots d^{(Z)}_{x_Z''} \; \delta_{x_R'},$$

Kronecker deltas from contractions

$$\text{loop} = X^+_{p'} X_{p''} X^+_{q'} X_{q''} \ldots X^+_{v'} X_{v''} X^+_{w'} X_{w''}$$

and

open path $= X^+_{p'} X_{p''} X^+_{q'} X_{q''} \cdots X^+_{v'} X_{v''} X^+_{w'} X_{w''}$

Here we used the following observations:

$$X^+ X \cdots X^+ X \cdots X^+ X \cdots X^+ X \cdots X^+ X$$

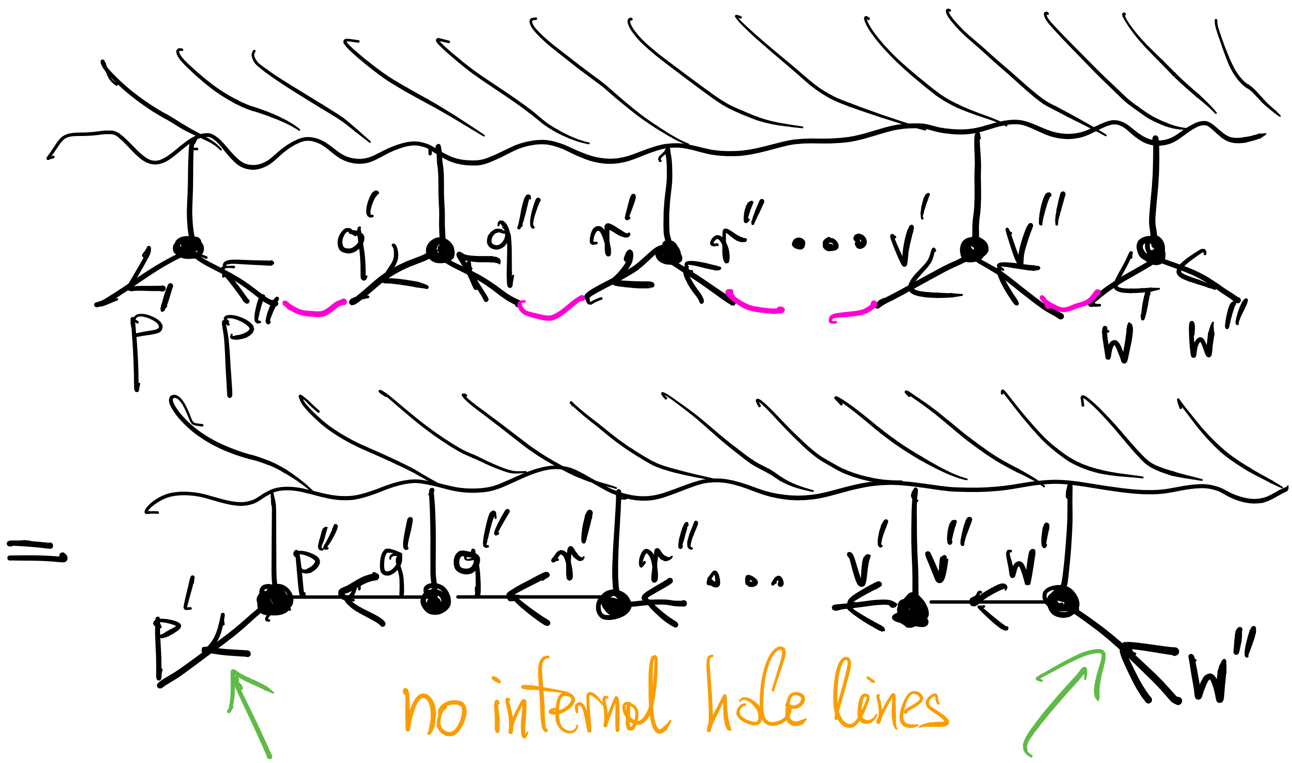
lead to open paths

$$X^+ X \cdots X^+ X \cdots X^+ X \cdots X^+ X \cdots X^+ X$$

lead to loops

Let us now "unscramble" open paths and loops. To do this, we consider 5 cases and the role of the number of internal hole lines in open paths and loops. We will start with open paths.

(I) [canonical open path]

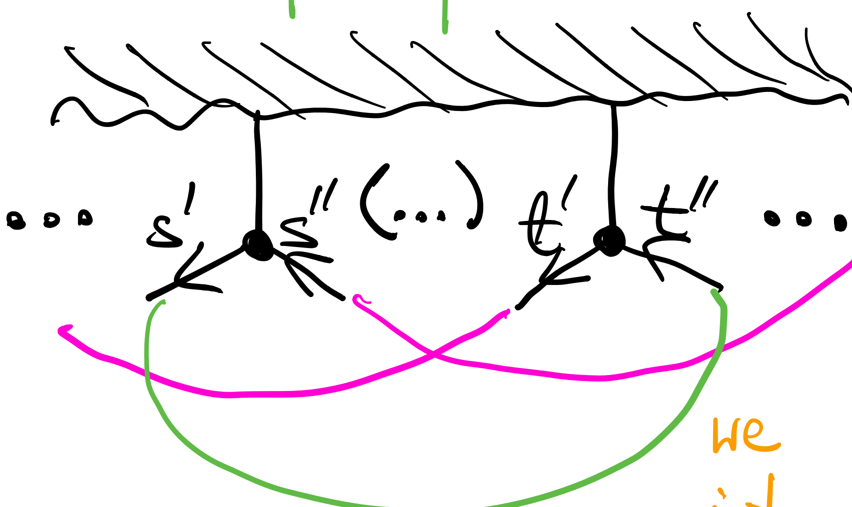


$$N\big[\cdots X^{\dagger}_{p'}\,\overbrace{X_{p''}X^{\dagger}_{q'}}\,\overbrace{X_{q''}X^{\dagger}_{r'}}\,\overbrace{X_{r''}\cdots X^{\dagger}_{v'}}\,\overbrace{X_{v''}X^{\dagger}_{w'}}\,X_{w''}\cdots\big]$$

$$= +\,\overbrace{X_{p''}X^{\dagger}_{q'}}\;\overbrace{X_{q''}X^{\dagger}_{r'}}\cdots\overbrace{X_{v''}X^{\dagger}_{w'}}$$

$$\times\, N\big[\cdots X^{\dagger}_{p'}\,X_{w''}\cdots\big]$$

(II)



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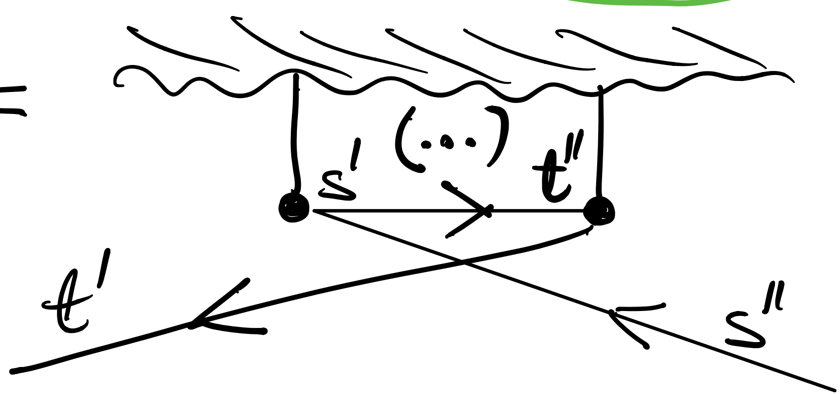


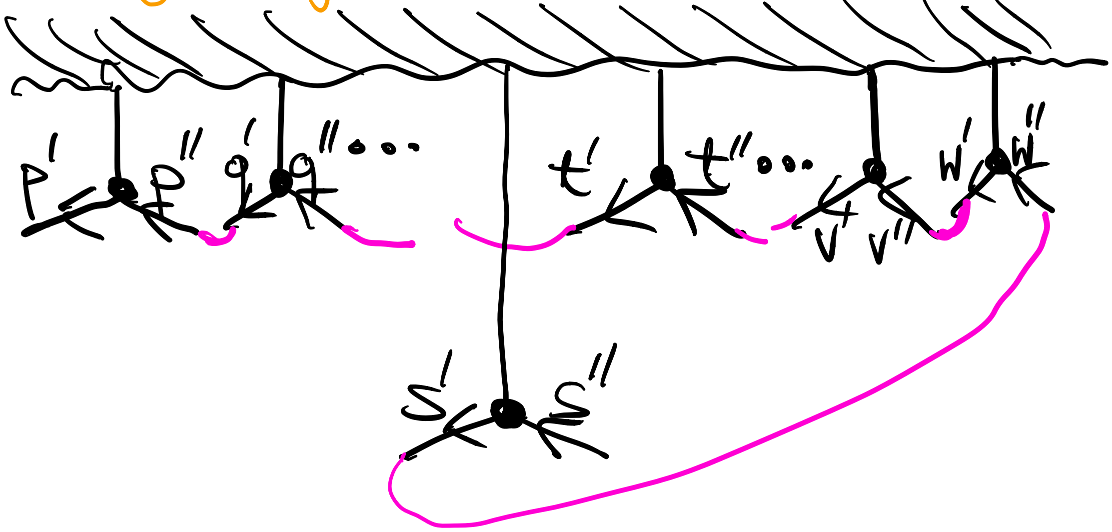
we introduced one internal hole line ($s't''$)

$$N[\dots X^{+}_{s'} X_{s''} \dots X^{+}_{t'} X_{t''} \dots] = -\, \underbrace{X^{+}_{s'} X_{t''}}_{\text{internal hole line}} N[\dots X^{+}_{t'} X_{s''} \dots]$$

sign change

internal hole line

III



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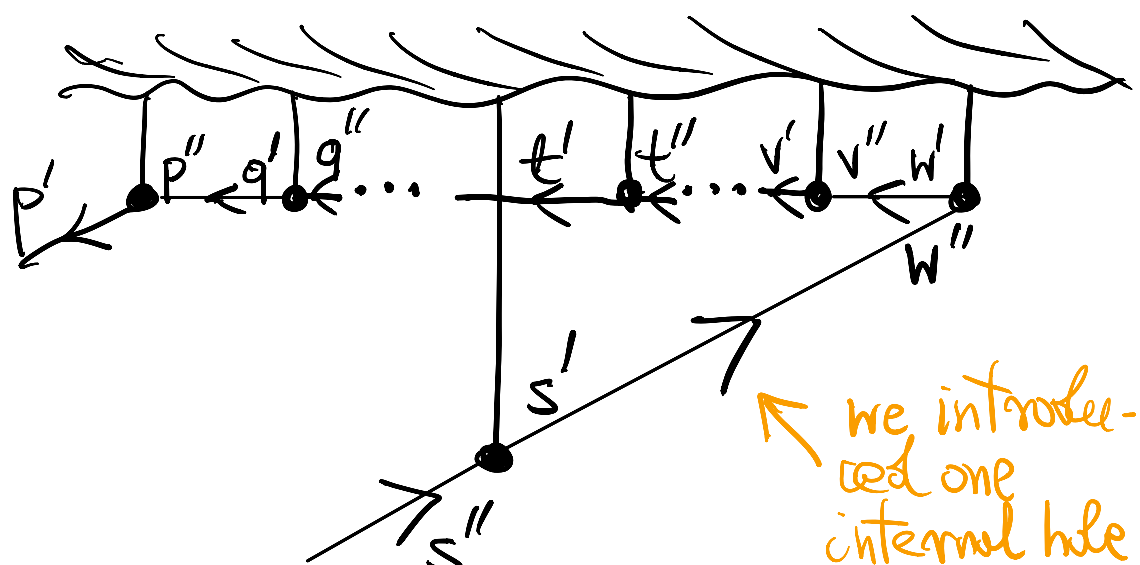


we introduced one internal hole line

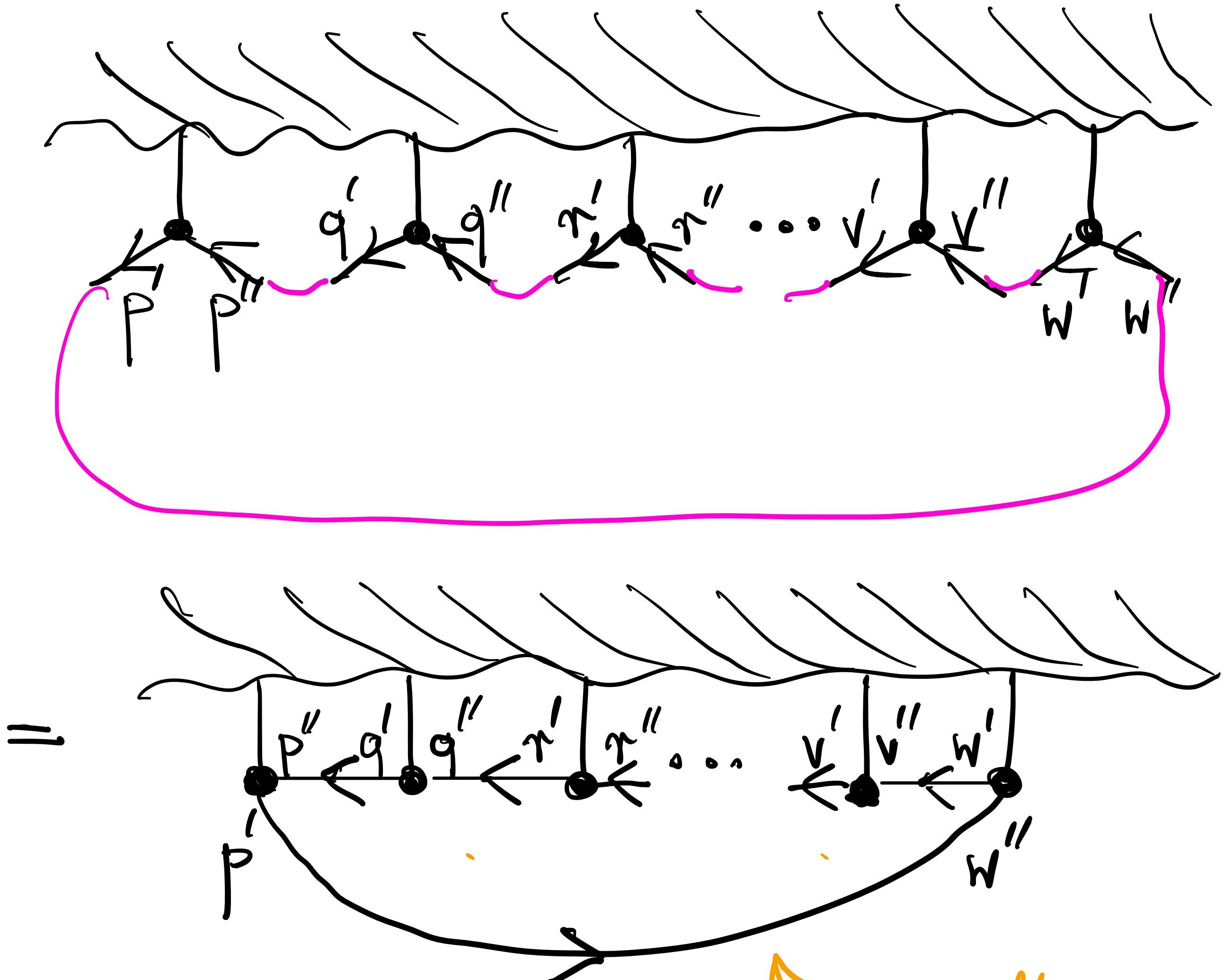
$$N[\dots X^\dagger_{p'} X_{p''} X^\dagger_{q'} X_{q''} \dots X^\dagger_{s'} X_{s''} X^\dagger_{t'} X_{t''} \dots X^\dagger_{v'} X_{v''} X^\dagger_{w'} X_{w''} \dots]$$

$$= - X_{p''} X^\dagger_{q'} \dots X_{v''} X^\dagger_{w'} \; \underbrace{X^\dagger_{s'} X_{w''}}_{\text{internal hole line}}$$

$$\times N[\dots X^\dagger_{p'} X_{s''} \dots]$$

Loops:

(IV) [canonical loop] (analog of I)



exactly one internal hole line (minimum for a loop)

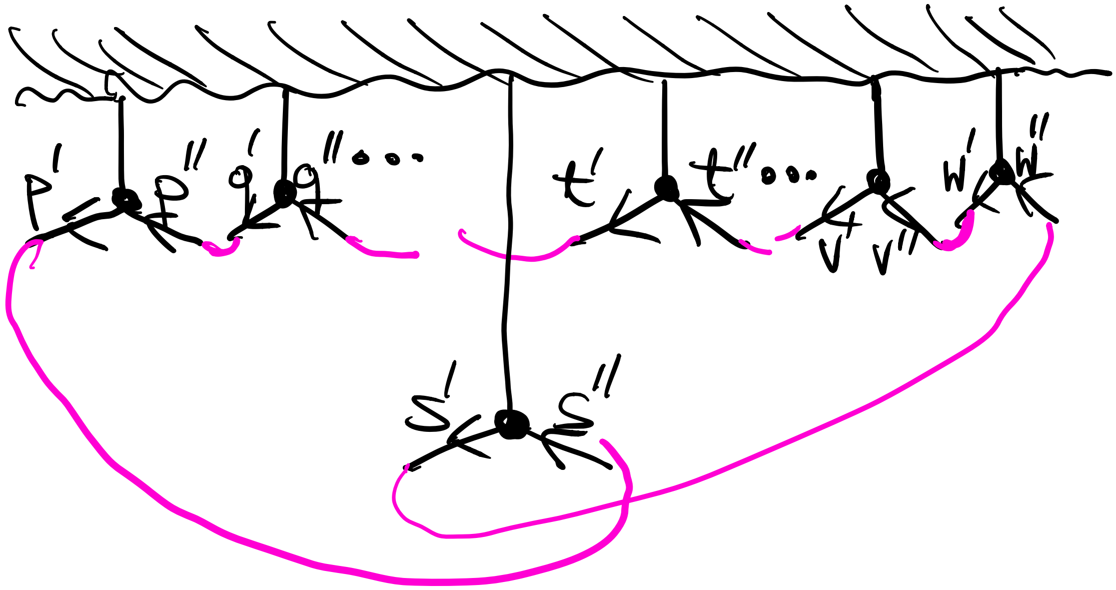
$$N[\cdots X^\dagger_{p'} X_{p''} X^\dagger_{q'} X_{q''} X^\dagger_{r'} X_{r''} \cdots X^\dagger_{v'} X_{v''} X^\dagger_{w'} X_{w''} \cdots]$$

$$= + X_{p''} X^\dagger_{q'} X_{q''} X^\dagger_{r'} \cdots X_{v''} X^\dagger_{w'}$$

$$\times \underbrace{X^\dagger_{p'} X_{w''}}_{\text{hole line}} N[\cdots]$$

Introduction of more internal hole lines can be done using case (II) ($\Delta h = 1$, sign will change) or an analog of (III), called (V).

(V)



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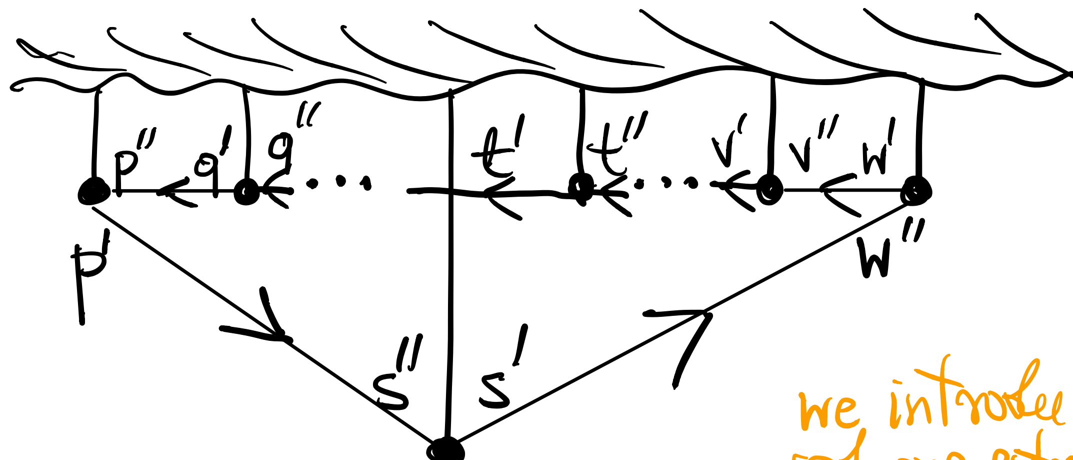


we introduced one extra internal hole line to canonical loop

$$N[\ldots X_{p'}^{\dagger} X_{p''} X_{q'}^{\dagger} X_{q''} \ldots X_{s'}^{\dagger} X_{s''} X_{t'}^{\dagger} X_{t''} \ldots X_{v'}^{\dagger} X_{v''} X_{w'}^{\dagger} X_{w''} \ldots]$$

$$= - X_{p''} X_{q'}^{\dagger} \ldots X_{v''} X_{w'}^{\dagger} \underbrace{X_{s'}^{\dagger} X_{w''}}_{\text{internal hole line}}$$

$$\times \underbrace{X_{p'}^{\dagger} X_{s''}}_{\text{internal hole line}} N[\ldots]$$

Summary:

(I) $h=0, +$

(II) $\Delta h=1, -$

(III) $\Delta h=1, -$

open paths

(IV) $h=1, +$

[(III)] $\Delta h=1, -$

(V) $\Delta h=1, -$

loops

$$N\big[(\mathrm{loop})_1 \cdots (\mathrm{loop})_{l_R} \, (\mathrm{open\ path})_1 \cdots (\mathrm{open\ path})_{m_R}\big]$$

$$= \delta_{\chi_R}' \, N\Big[\underbrace{X^{\dagger}_{p_1^{(R)}} X_{q_1^{(R)}}}_{\text{external lines of (open path)}_1} \cdots \underbrace{X^{\dagger}_{p_{m_R}^{(R)}} X_{q_{m_R}^{(R)}}}_{\text{external lines of (open path)}_{m_R}}\Big]$$

$$\times \; s_R \, , \quad \text{where}$$

$$s_R = \prod_{\lambda=1}^{l_R} (-1)^{h_\lambda + 1} \prod_{\mu=1}^{m_R} (-1)^{h_\mu} ,$$

with

$$h_\lambda = \#\text{ of internal hole lines in } (\text{loop})_\lambda$$

and

$$h_\mu = \#\text{ of internal hole lines in } (\text{open path})_\mu .$$

We obtain

$$s_R = (-1)^{l_R + h_R}$$

$$h_R = \sum_{\lambda=1}^{l_R} h_\lambda + \sum_{\mu=1}^{m_R} h_\mu$$

$$= \#\text{ of all internal hole lines in } R .$$

Thus,

$$K_A \cdots K_Z = \sum_R K_R \,, \text{ where}$$

$$K_R = s_R h_R \sum_{\chi_R', \chi_R''} d^{(R)}_{\chi_R', \chi_R''}$$

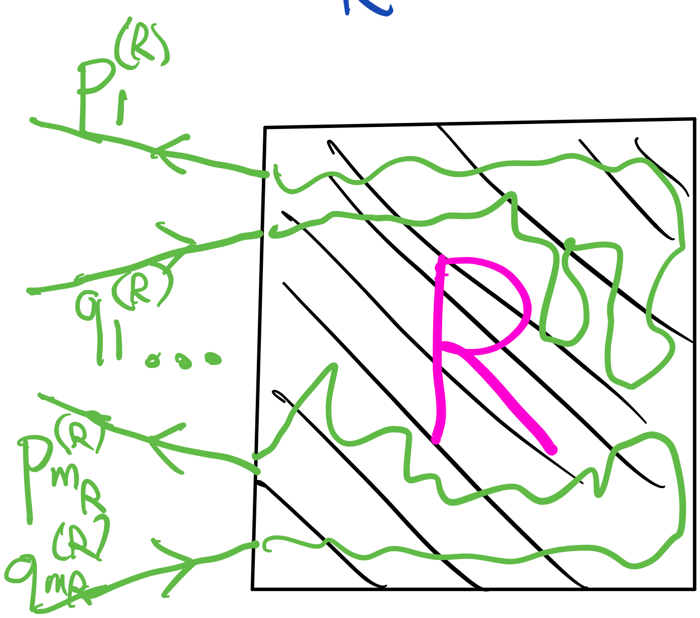
product of scalars associated with $A, \dots, Z$

$$\times N\left[\prod_{r=1}^{m_R} X^{\dagger}_{p_r^{(R)}} X_{q_r^{(R)}} \right]$$

and

$$s_R = (-1)^{l_R + h_R} \,.$$

χ_R'' consists of $p_r^{(R)}, q_r^{(R)}$, $r = 1, \dots, m_R$.



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Comments:
How about standard operator ordering?

$$K_C = w_C \sum_{\chi_C''} d^{(C)}_{\chi_C''} \hat{O}_{\chi_C''},$$

where

$$\hat{O}_{\chi_C''} = X^\dagger_{p_1^{(C)}} \cdots X^\dagger_{p_g^{(C)}} X_{q_g^{(C)}} \cdots X_{q_1^{(C)}}$$

We now allow contractions within each operator K_C written in a standard form. These contractions are

$$X_p^\dagger X_q = \eta(p)\, \delta_{pq},$$

meaning that the resulting internal lines are HOLE lines.

The theorem and the rules remain the same as in the case of normal-ordered operators $K_A, \ldots, K_Z$, but we allow contractions originating from the same vertex and label the corresponding internal lines with hole indices $(i, j, k, \ldots)$.

Situations of common interest:

$$K_A \ldots K_Z |\Phi\rangle,$$

$$\langle\Phi| K_A \ldots K_Z |\Phi\rangle.$$

$$\langle \Phi | K_A \cdots K_Z | \Phi \rangle = \sum_{\substack{R \\ \text{f.c.}}} \langle \Phi | K_R | \Phi \rangle$$

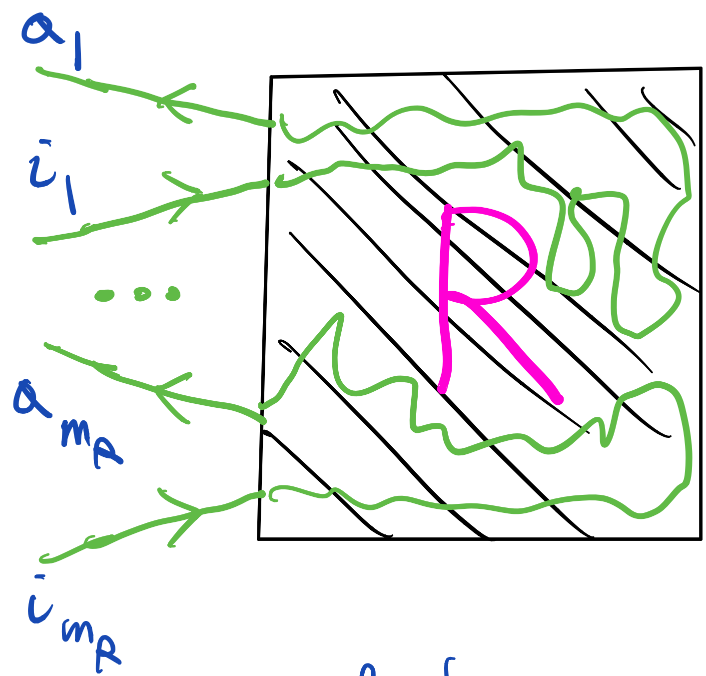
[NO EXTERNAL LINES]

$$K_R = s_R w_R \sum_{\chi_R'} d^{(R)}_{\chi_R'} =$$

$$= (-1)^{l_R + h_R} w_R \sum_{\chi_R'} d^{(R)}_{\chi_R'} .$$

$$K_A \cdots K_Z | \Phi \rangle = \sum_{R,\, \text{a.f.c.}} K_R | \Phi \rangle$$

[ALL EXTERNAL LINES MUST EXTEND TO THE LEFT]



$$K_R|\Phi\rangle = (-1)^{l_R + h_R} w_R \times \sum_{\chi_R'} \sum_{\substack{a_1\dots a_{m_R}\\ i_1\dots i_{m_R}}} d^{(R)}_{\chi_R',\, a_1\dots a_{m_R},\, i_1\dots i_{m_R}} \times \left|\Phi^{a_1\dots a_{m_R}}_{i_1\dots i_{m_R}}\right\rangle$$

$$\left(\prod_{r=1}^{m_R} X^\dagger_{a_r} X_{i_r} |\Phi\rangle\right)$$

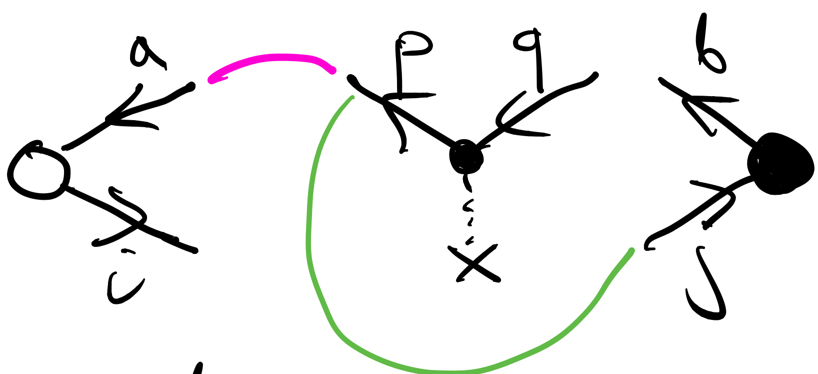
Examples:

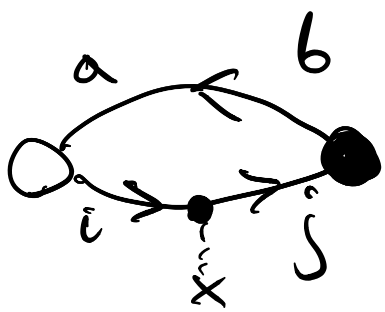
$$\langle \Phi_i^a | Z_N | \Phi_j^b \rangle$$

$$= \langle \Phi | (E_i^a)^\dagger Z_N E_j^b | \Phi \rangle$$

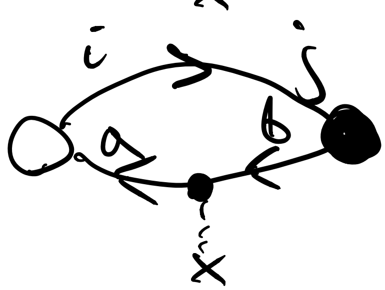
$$(E_i^a = N[X_a^\dagger X_i])$$

p,q-free
a,i,b,j-fixed





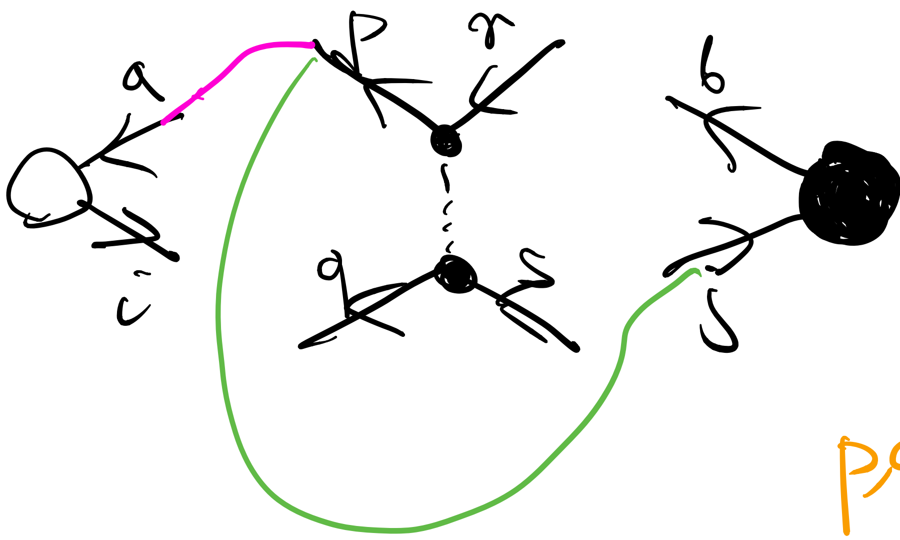
$w_R = 1,\ s_R = (-1)^{1+2} = -1$



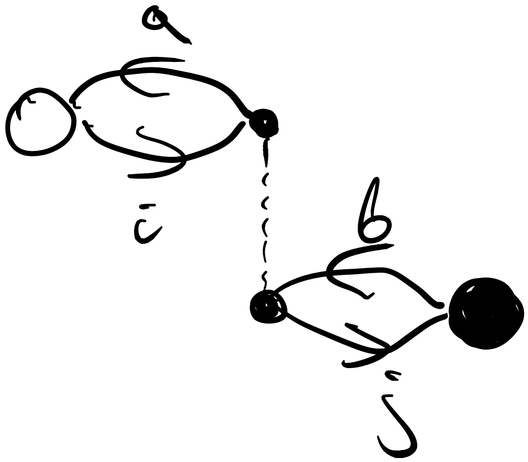
$w_R = 1,\ S_R = (-1)^{1+1} = +1$

$$\langle \Phi_i^a | Z_N | \Phi_j^b \rangle = -\langle j | \hat{z} | i \rangle \delta_{ab} + \langle a | \hat{z} | b \rangle \delta_{ij}$$

$$\langle \Phi_i^a | V_N | \Phi_j^b \rangle = \langle \Phi | (E_i^a)^\dagger V_N E_j^b | \Phi \rangle$$

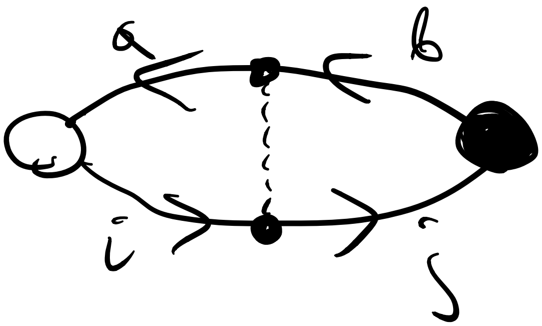


p,q, r,s - free
a,i,b,j - fixed



$w_R = 1,$

$s_R = (-1)^{2+2} = +1$



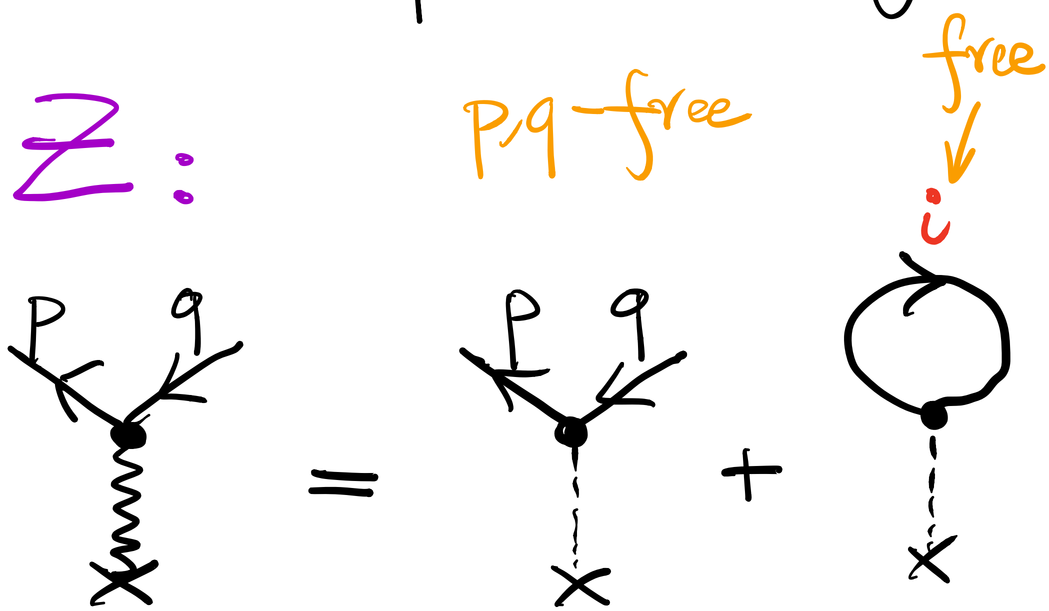
$w_R = 1,$

$s_R = (-1)^{1+2} = -1$

$$\langle \Phi_i^a | V_N | \Phi_j^b \rangle = \langle aj | \hat{v} | ib \rangle - \langle aj | \hat{v} | bi \rangle = \langle aj | \hat{v} | ib \rangle_A$$

Relationships between Z and Z_N and V and V_N diagrammatically (to practice diagrams originating from standard operator ordering).

Z:



$w_R = 1$ $\quad$ $w_R = 1$

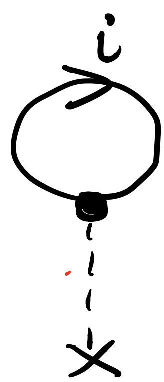
$S_R = (-1)^{0+0} = +1$ $\quad$ $S_R = (-1)^{1+1} = +1$

$(h_R = 0, \ell_R = 0)$ $\quad$ $(h_R = 1, \ell_R = 1)$

$$Z = \sum_{pq} \langle p|\hat{z}|q\rangle N[X_p^\dagger X_q]$$

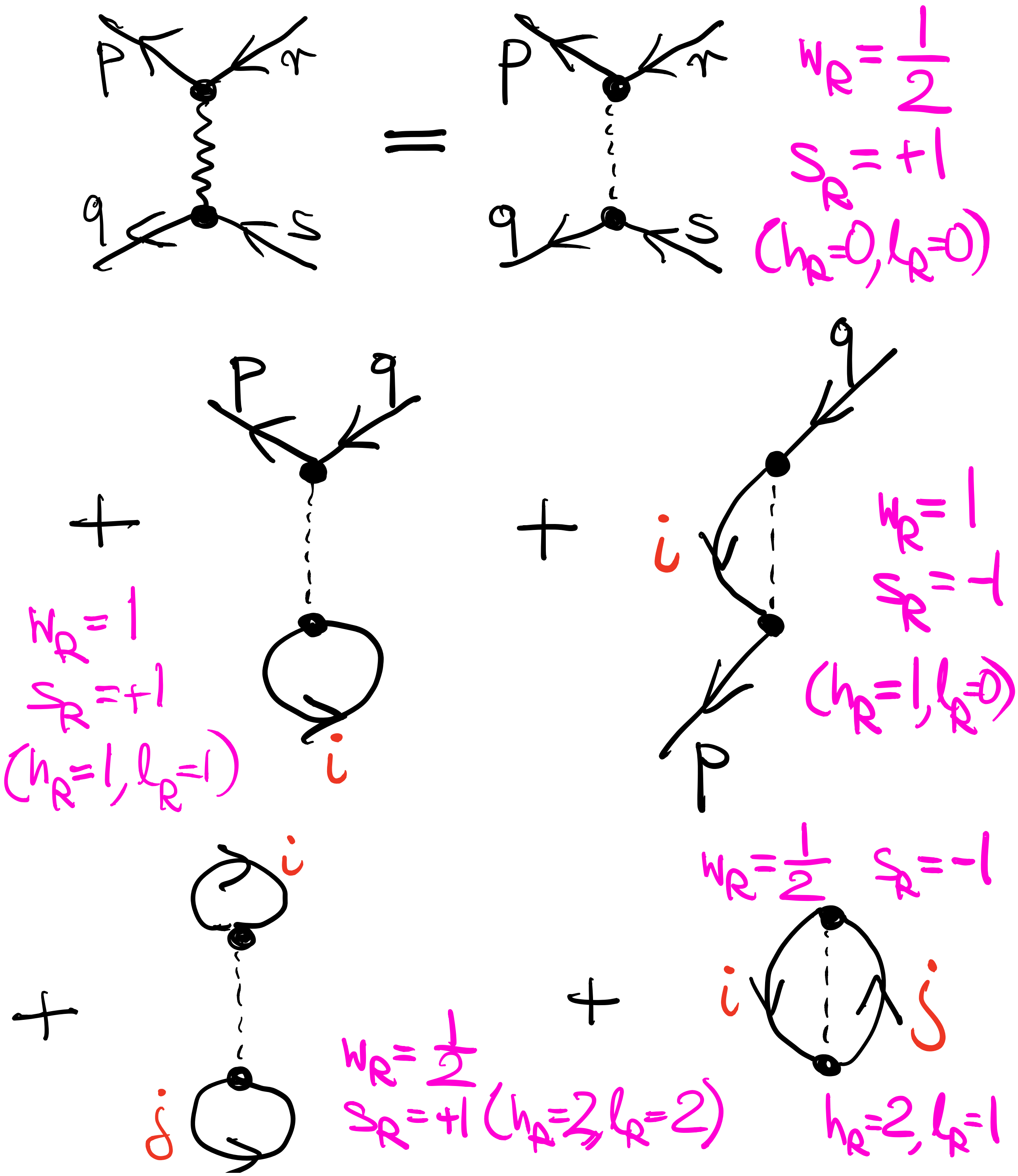
$$+ \sum_i \langle i|\hat{z}|i\rangle$$

$$= Z_N + \langle \Phi|Z|\Phi\rangle$$



V:

p, q, r, s - free



p
r
q
s
=
$W_R = \frac{1}{2}$
$S_R = +1$
$(h_R = 0, l_R = 0)$
+
$W_R = 1$
$S_R = +1$
$(h_R = 1, l_R = 1)$
i
+
$W_R = 1$
$S_R = -1$
$(h_R = 1, l_R = 0)$
+
i
j
$W_R = \frac{1}{2}$
$S_R = +1$ $(h_R = 2, l_R = 2)$
+
$W_R = \frac{1}{2}$ $S_R = -1$
$h_R = 2, l_R = 1$

$$V = \frac{1}{2}\sum_{pq,r,s} \langle pq|\hat{v}|rs\rangle N[X_p^\dagger X_r X_q^\dagger X_s]$$

$$+ \sum_{pq}\sum_{i} \langle pi|\hat{v}|qi\rangle N[X_p^\dagger X_q]$$

$$- \sum_{pq}\sum_{i} \langle pi|\hat{v}|iq\rangle N[X_p^\dagger X_q]$$

$$+ \frac{1}{2}\sum_{i,j} \langle ij|\hat{v}|ij\rangle$$

$$- \frac{1}{2}\sum_{i,j} \langle ij|\hat{v}|ji\rangle$$

$$V = V_N + \sum_{pq}\sum_{i} \langle pi|\hat{v}|qi\rangle_A N[X_p^\dagger X_q]$$

$$+ \frac{1}{2}\sum_{ij} \langle ij|\hat{v}|ij\rangle_A$$

$$= V_N + G_N + \langle \Phi | V | \Phi \rangle$$

$$G_N = \sum_{pq} \langle p | g' | q \rangle N[X_p^\dagger X_q]$$

$$\langle p | g' | q \rangle = \sum_{i} \langle pi | v' | qi \rangle_A$$

$$\langle \Phi | V | \Phi \rangle = \frac{1}{2} \sum_{ij} \langle ij | v' | ij \rangle_A$$