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MANY-BODY PERTURBATION THEORY (MBPT)

MBPT is an application of the Rayleigh-Schrödinger perturbation theory ← RSPT to the many-fermion Schrödinger equation,

$$H|\Psi_0\rangle = E_0|\Psi_0\rangle,$$

for the non-degenerate ground-state $|\Psi_0\rangle$.

$$|\Psi_0\rangle = |\Psi_0^{(0)}\rangle + |\Psi_0^{(1)}\rangle + \dots = \sum_{n=0}^{\infty} |\Psi_0^{(n)}\rangle$$

$$E_0 = E_0^{(0)} + E_0^{(1)} + E_0^{(2)}$$
$$+ \cdots$$

We assume that

IPM state

$$|\Psi_0^{(0)}\rangle = |\Phi_0\rangle$$

$$\left(|\Psi_0\rangle \approx |\Phi_0\rangle \right) .$$

$|\Phi_0\rangle$ will also be a Fermi vacuum.

We will rewrite the Schrödinger equation as follows:

$$H_N |\Psi_0\rangle = \Delta E_0 |\Psi_0\rangle ,$$

where

$$H_N = H - \langle \Phi_0 | H | \Phi_0 \rangle ,$$
$$\Delta E_0 = E_0 - \langle \Phi_0 | H | \Phi_0 \rangle .$$

If $|\Phi_0\rangle$ is a HF state,
then

$$\Delta E_0 = E_0 - \langle\Phi_0|H|\Phi_0\rangle$$
$$= E_0 - E_0^{(HF)}$$
$$= \text{correlation energy} \quad \checkmark$$

In this class,

$$\Delta E_0 = E_0 - \langle\Phi_0|H|\Phi_0\rangle$$

will be called the correlation energy independent of the type of the IPM state $|\Phi_0\rangle$.

$$|\Psi_0\rangle = |\Psi_0^{(0)}\rangle + |\Psi_0^{(1)}\rangle + \cdots = \sum_{n=0}^{\infty} |\Psi_0^{(n)}\rangle$$

$$\Delta E_0 = \Delta E_0^{(0)} + \Delta E_0^{(1)} + \Delta E_0^{(2)} + \cdots = \sum_{n=0}^{\infty} \Delta E_0^{(n)}.$$

Before talking about MBPT, we review RSPT for

$$K|\Psi_0\rangle = k_0|\Psi_0\rangle.$$

(eventually, $K = H_N$, $k_0 = \Delta E_0$).

We assume that K can be split as follows:

unperturbed operator

perturbation ($W = K - K_0$)

$$K = K_0 + W,$$

where we know how to solve the unperturbed problem,

$$K_0|\Phi_n\rangle = \varkappa_n|\Phi_n\rangle,$$

$$n = 0, 1, 2, \ldots$$

$$\langle\Phi_n|\Phi_m\rangle = \delta_{nm}.$$

$|\Phi_n\rangle$'s form a basis in the Hilbert space.

In RSPT,

$$|\Psi_0\rangle = \overbrace{\Omega}^{\text{wave operator}} |\Phi_0\rangle , \quad \checkmark$$

$$\langle\Phi_0|\Psi_0\rangle = \langle\Phi_0|\Omega|\Phi_0\rangle = \langle\Phi_0|\Phi_0\rangle = 1$$

intermediate normalization

(we will not normalize $|\Psi_0\rangle$ to unity,

$$\cancel{\langle\Psi_0|\Psi_0\rangle = \langle\Phi_0|\Omega^\dagger\Omega|\Phi_0\rangle = 1}\;)$$

Preliminary steps:

$$\langle \Phi_0| \,/\, (K_0 + W)|\Psi_0\rangle = k_0|\Psi_0\rangle$$

$$\underbrace{\langle \Phi_0|K_0}_{\kappa_0\langle\Phi_0|}|\Psi_0\rangle + \langle \Phi_0|W|\Psi_0\rangle = k_0 \underbrace{\langle\Phi_0|\Psi_0\rangle}_{=1}$$

$$\kappa_0 + \langle\Phi_0|W\underbrace{|\Psi_0\rangle}_{\Omega|\Phi_0\rangle} = k_0$$

$$k_0 = \kappa_0 + \langle\Phi_0|\tau|\Phi_0\rangle,$$

where

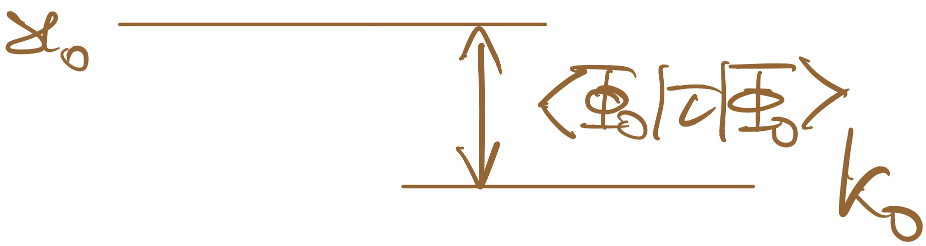
$$\tau = PW\Omega \;(= PW\Omega P),$$

$$P = |\Phi_0\rangle\langle\Phi_0|$$

(we used $P|\Phi_0\rangle = |\Phi_0\rangle$)

$$K_0 = \varepsilon_0 + \langle \Phi_0 | \tau | \Phi_0 \rangle$$

$$\tau = PW\Omega \leftarrow \text{reaction operator}$$



RSPT provides us with a specific design of Ω and τ (in particular, $\Omega|\Phi_n\rangle$, $n = 1, 2, \ldots,$ is zero).

In order to formulate RSPT, we must introduce REDUCED RESOLVENTs of K and K_0.

We will follow Per-Olov Löwdin.

dimension of the Hilbert space $\mathcal{H}$

$$P = |\Phi_0\rangle\langle\Phi_0|$$

$$Q = \mathbb{1} - P = \sum_{n=1}^{(\infty)} |\Phi_n\rangle\langle\Phi_n|$$

$$\mathcal{H} = \mathcal{H}_P \oplus \mathcal{H}_Q$$

$$\mathcal{H}_P = P\mathcal{H}$$

$$\mathcal{H}_Q = Q\mathcal{H}$$

$$P^2 = P, \quad Q^2 = Q$$

$$P^\dagger = P, \quad Q^\dagger = Q$$

$$PQ = QP = 0$$

$$\mathcal{Q}_x(K) = Q\left[\underset{\substack{\uparrow \\ \text{arbitrary number} \in \mathbb{R},\ \neq 0}}{\alpha} P + Q(\overset{\substack{x \in \mathbb{R} \\ \downarrow}}{x} - K)Q\right]^{-1} Q$$

Let's first understand what is $\mathcal{Q}_x(K)$:

$$A \xrightarrow[\text{representation}]{\text{matrix}} \begin{array}{cc|l} \overbrace{\qquad}^{|\Phi\rangle} & \overbrace{\qquad}^{|\Phi_n\rangle,\ n \geq 1} & \\ \left(\begin{array}{c|c} A_{PP} & A_{PQ} \\ \hline A_{QP} & A_{QQ} \end{array}\right. & \left.\right) & \begin{array}{l} \big\} |\Phi\rangle \\ \big\} |\Phi_n\rangle,\ n \geq 1 \end{array} \end{array}$$

$$P \Rightarrow \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$Q \Rightarrow \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_Q \end{array}\right)$$

$\boxed{\begin{array}{l}\mathbb{1}_Q \text{ - the} \\ \text{unit matrix} \\ \text{in the} \\ Q \text{ space}\end{array}}$

$$K \Rightarrow \left(\begin{array}{c|c} K_{PP} & K_{PQ} \\ \hline K_{QP} & K_{QQ} \end{array}\right)$$

$$K_{PP} = PKP \;, \quad K_{QP} = QKP$$

$$K_{PQ} = PKQ \;, \quad K_{QQ} = QKQ$$

$$[\alpha P + Q(x - K)Q]$$

$$= \left(\begin{array}{c|c} \alpha & 0 \\ \hline 0 & 0 \end{array}\right) +$$

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_Q \end{array}\right)\left(\begin{array}{c|c} x1 - K_{PP} & -K_{PQ} \\ \hline -K_{QP} & x\mathbb{1}_Q - K_{QQ} \end{array}\right)\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_Q \end{array}\right)$$

$$= \left(\begin{array}{c|c} \alpha & 0 \\ \hline 0 & x\mathbb{1}_Q - K_{QQ} \end{array}\right) \quad \checkmark$$

$$\mathcal{R}_x(K) =$$

$$= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_Q \end{array}\right)\left(\begin{array}{c|c} \alpha^{-1} & 0 \\ \hline 0 & (x\mathbb{1}_Q - K_{QQ})^{-1} \end{array}\right)$$

$$\cdot \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_Q \end{array}\right)$$

$$= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (x\mathbb{1}_Q - K_{QQ})^{-1} \end{array}\right)$$

no α

$$A \Longrightarrow (\lambda \mathbb{1} - A)^{-1}$$

RESOLVENT of A

Properties of $R_x(K)$:

(1) $R_x = R_x^{\dagger}$ $(K = K^{\dagger})$
$(x \in \mathbb{R})$

(2) $P R_x = R_x P = 0$
$(PQ = QP = 0)$

(3) $$\frac{\partial R_x}{\partial \alpha} = 0$$

(4) $$Q(x-K)R_x = Q$$

$$\left(R_x(x-K)Q = Q \right)$$ ✓

(5) $$\frac{\partial R_x}{\partial x} = -R_x^2$$

Proofs of (3)–(5):

$$[\alpha P + Q(x-K)Q_1][\alpha P + Q(x-K)Q_1]^{-1} = 1$$

To prove (4), we multiply this by Q on the left and by Q on the right:

$$Q[\alpha P+Q(\mathcal{X}-K)Q][\alpha P+Q(\mathcal{X}-K)Q]^{-1}Q$$
$$= QQ = Q$$

$$Q(\mathcal{X}-K)\underbrace{Q[\alpha P+Q(\mathcal{X}-K)Q]^{-1}Q}_{R_{\mathcal{X}}(K)}$$
$$= Q$$

$$Q(\mathcal{X}-K)R_{\mathcal{X}}(K) = Q$$

To prove (3), we perform the following:

$$P[\alpha P+Q(\mathcal{X}-K)Q][\alpha P+Q(\mathcal{X}-K)Q]^{-1}Q$$
$$= PQ = 0$$

$$\alpha P[\alpha P+Q(\mathcal{X}-K)Q]^{-1}Q$$
$$= 0 \quad , \alpha \neq 0$$

$$P\left[\alpha P + Q(x-K)Q\right]^{-1} Q = 0$$

$$\frac{\partial R_x}{\partial \alpha} = Q\left(\frac{\partial}{\partial \alpha}\left[\alpha P + Q(x-K)Q\right]^{-1}\right)Q$$

In general,

$$A = A(\alpha)$$

$$AA^{-1} = \mathbb{1}$$

$$\frac{\partial}{\partial \alpha}\left(AA^{-1}\right) = 0$$

$$\frac{\partial A}{\partial \alpha} A^{-1} + A \frac{\partial A^{-1}}{\partial \alpha} = 0$$

$$\frac{\partial A^{-1}}{\partial \alpha} = -A^{-1} \frac{\partial A}{\partial \alpha} A^{-1} \checkmark$$

$$\frac{\partial R_x}{\partial x} = -Q\left[\alpha P + Q(x-K)Q\right]^{-1}$$

$$\times \underbrace{P\left[\alpha P + Q(x-K)Q\right]^{-1}Q}_{=0}$$

$$= 0.$$

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(5):

$$\frac{\partial R_x}{\partial x} = Q\left(\frac{\partial}{\partial x}\left[\alpha P + Q(x-K)Q\right]^{-1}\right)Q$$

$$= -Q\left[\alpha P + Q(x-K)Q\right]^{-1}Q$$

$$\times Q\left[\alpha P + Q(x-K)Q\right]^{-1}Q$$

$$= -R_{\mathcal{X}}^{2}$$

$\therefore$

Because of (4), we often write

$$R_{\mathcal{X}}(K) = \frac{Q}{\mathcal{X} - K} \quad \checkmark$$

symbolic

$$Q(\mathcal{X}-K)R_{\mathcal{X}} = Q$$

$R_{\mathcal{X}}(K)$ has singularies for $\mathcal{X}$ values being the eigen-values of

$$K_{QQ} = QKQ \,.$$

Back to foundations of RSPT:

$$K|\Psi_0\rangle = k_0|\Psi_0\rangle$$

$$(k_0 - K)|\Psi_0\rangle = 0 \quad \checkmark$$

$$\Downarrow$$

$$(\varkappa - K)|\Psi_\varkappa\rangle = \ldots$$

function of $\varkappa$ that goes to 0 when $\varkappa = k_0$ ($\varkappa = k_n$)

Trial function:

$$|\Psi_\varkappa\rangle = \underbrace{|\Phi_0\rangle}_{\text{P-space component}} + \underbrace{Q_\varkappa K |\Phi_0\rangle}_{\text{Q-space component}}$$

$$\langle \Phi_0 | \Psi_x \rangle = \langle \Phi_0 | \Phi_0 \rangle = 1$$

$$\left(\langle \Phi_0 | R_x = 0, \quad \langle \Phi_0 | Q = 0 \right)$$

$$(x - K)|\Psi_x\rangle = P(x-K)|\Psi_x\rangle + Q(x-K)|\Psi_x\rangle$$

$$Q(x-K)|\Psi_x\rangle =$$

$$= Q(x-K)|\Phi_0\rangle + \underbrace{Q(x-K)R_x}_{Q \text{ from (4)}} K|\Phi_0\rangle$$

$$Q|\Phi_0\rangle = 0$$

$$= -QK|\Phi_0\rangle + QK|\Phi_0\rangle = 0$$

$$(x-K)|\Psi_x\rangle = P(x-K)|\Psi_x\rangle$$

$$= |\Phi_0\rangle\langle\Phi_0|(x-K)|\Psi_x\rangle$$

$$= x\,|\Phi_0\rangle\underline{\langle\Phi_0|\Psi_x\rangle} \leftarrow 1$$

$$- |\Phi_0\rangle\underbrace{\langle\Phi_0|K|\Psi_x\rangle}_{f(x)}$$

$$= [x - f(x)]\,|\Phi_0\rangle$$

✓ $$(x-K)|\Psi_x\rangle = (x-f(x))|\Phi_0\rangle$$

$$|\Psi_x\rangle = |\Phi_0\rangle + Q_x K|\Phi_0\rangle$$

$$f(x) = \langle\Phi_0|K|\Psi_x\rangle$$

$$= \langle \Phi_0 | K + K Q_x K | \Phi_0 \rangle$$

$f(x)$ — bracketing function

By solving $f(x) = x$,
we can obtain the eigenvalues and eigenstates of K,

$$(k_n - K)|\Psi_n\rangle = 0,$$

where

$$f(k_n) = k_n \quad \text{(fixed points of } f(x)\text{)},$$

$$|\Psi_n\rangle = |\Psi_{k_n}\rangle = |\Phi_0\rangle + Q_{k_n} K |\Phi_0\rangle$$

(nonzero $|\Phi_0\rangle$ component)

We can use the above equations to construct only those solutions of the eigenvalue problem for K for which k_ns are not the eigenvalues of K_{QQ} and $|\Psi_n\rangle$s are not orthogonal to $|\Phi_0\rangle$. k_0 and $|\Psi_0\rangle$ are among them,

unperturbed

$$|\Psi_0\rangle = |\Phi_0\rangle + R_{k_0} K |\Phi_0\rangle$$

$$k_0 = f(k_0) = \langle \Phi_0 | K + K R_{k_0} K | \Phi_0 \rangle$$

$$|\Psi_0\rangle = \Omega |\Phi_0\rangle \quad (R_{k_0} K P \, \text{ex})$$

$$\Omega = P + R_{k_0} K P$$

$$k_0 = \langle \Phi_0 | K \Omega | \Phi_0 \rangle$$

$$(\Omega^{(0)} = P)$$

Generalization to arbitrary x (other than singularities of Q_x):

$$|\Psi_x\rangle = \Omega_x |\Phi_0\rangle$$

$$\Omega_x = P + Q_x K P$$

$$f(x) = \langle \Phi_0 | K \Omega_x | \Phi_0 \rangle$$

$$(x - K)|\Psi_x\rangle = [x - f(x)]|\Phi_0\rangle$$

Why do we call $f(x)$ a bracketing function?

$$f(x) = \langle \Phi_0 | K + K Q_x K | \Phi_0 \rangle$$

$$\lim_{x \to \pm\infty} f(x) = \langle \Phi_0 | K | \Phi_0 \rangle$$

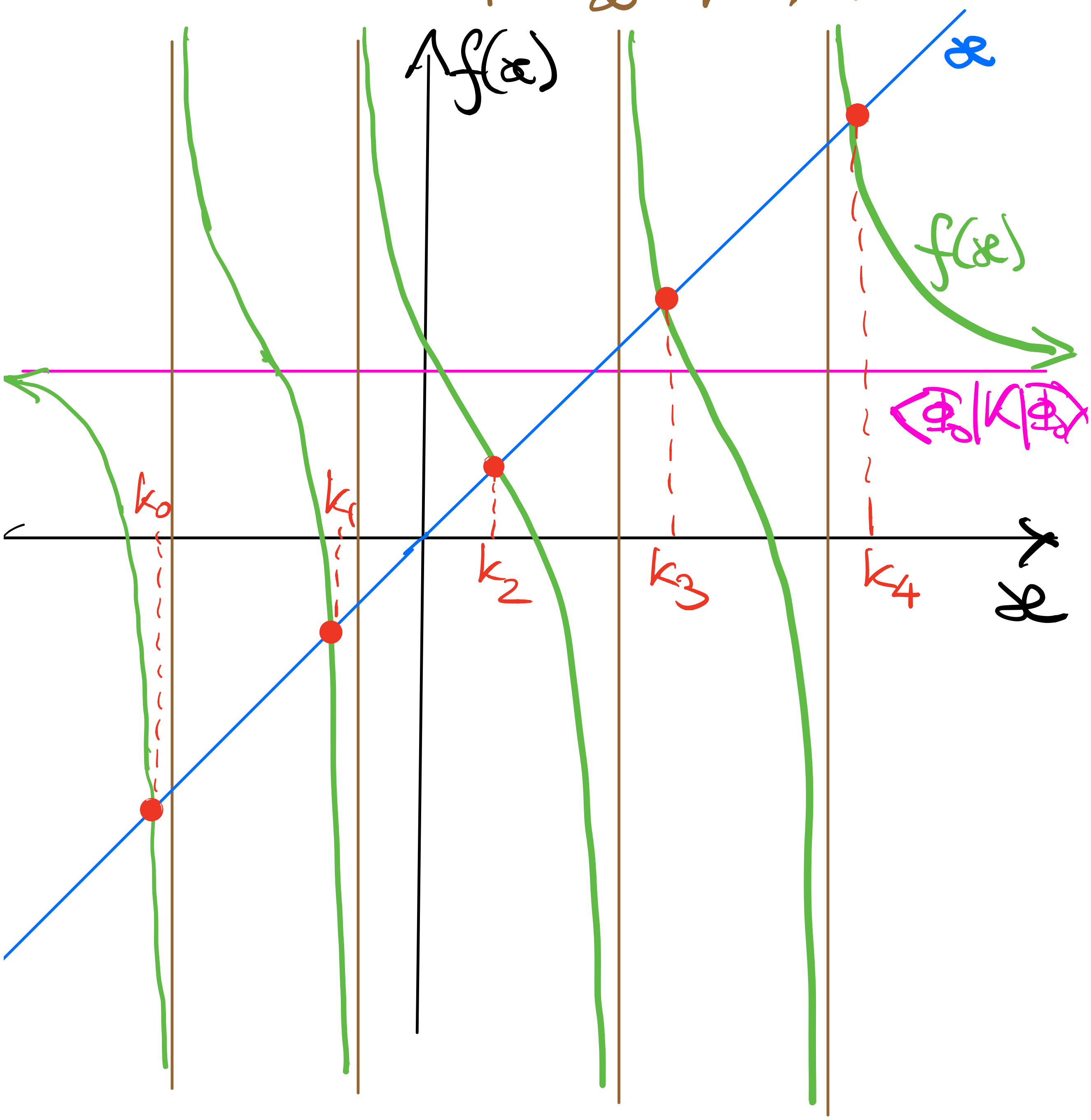
$$\frac{df}{dx} = \frac{d}{dx} \langle \Phi_0 | K + K Q_x K | \Phi_0 \rangle$$

$$= \langle \Phi_0 | K \frac{dQ_x}{dx} K | \Phi_0 \rangle$$

$$\overset{(5)}{=} - \langle \Phi_0 | K Q_x^2 K | \Phi_0 \rangle$$

$$= - \langle \Phi_0 | \underbrace{K Q_x}_{(Q_x K)^\dagger} Q_x K | \Phi_0 \rangle$$

$$= - \| Q_x K | \Phi_0 \rangle \|^2 < 0$$



$(x, f(x))$ brackets at least one eigenvalue of K.

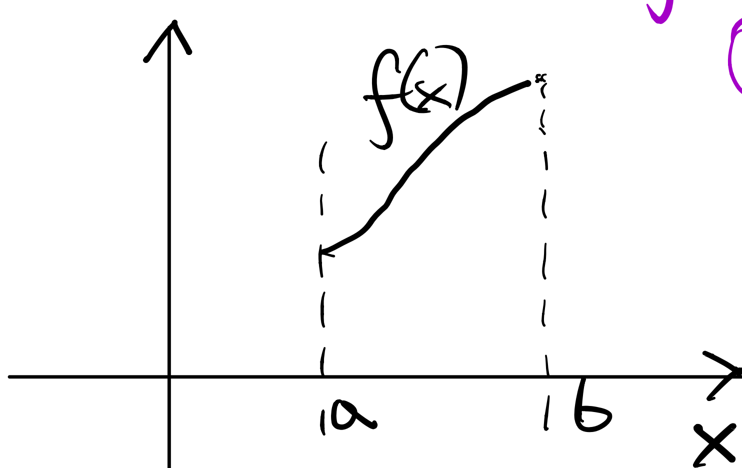
Let us write x as

$$x = k_n + \varepsilon$$ ✓

$$f(x) - k_n = f(x) - f(k_n)$$

$$= f(k_n + \varepsilon) - f(k_n) = f'(k_n + \theta\varepsilon)\cdot\varepsilon$$

$$0 < \theta < 1$$



$$\frac{f(b) - f(a)}{b - a} = f'(a + \theta(b - a))$$

$$0 < \theta < 1$$

$$f'(k_n+\Theta\varepsilon)<0$$

If $\varepsilon>0$, $f(x)-k_n<0$

If $\varepsilon<0$, $f(x)-k_n>0$

We can think of using $f(x)$ to determine lower bounds to eigenvalues of K.

The fastest proof of variational principle:

$\varkappa = -\infty$: $(-\infty, f(-\infty))$ brackets at least eigenvalue of K

Thus, there exists k_n such that

$$-\infty < k_n < f(-\infty)$$
$$= \langle \Phi_0 | K | \Phi_0 \rangle$$

$$\langle \Phi_0 | K | \Phi_0 \rangle > k_n \geq k_0$$

$\therefore$

Properties of $\Omega_\varkappa$ (or Ω):

(i) $P\Omega_\varkappa = P$

(intermediate normalization condition)

(ii) $\Omega_x P = \Omega_x$

(iii) $\Omega_x^2 = \Omega_x$.

(but $\Omega_x^\dagger \neq \Omega_x$)

Proofs:

(i) $P\Omega_x = P(P + Q_x K P)$

$\overset{(2)}{=} P^2 = P$ ✓

(ii) $\Omega_x P = (P + Q_x K P) P$

$= P^2 + Q_x K P^2$

$= P + Q_x K P = \Omega_x$

(iii) $\Omega_x^2 \overset{(ii)}{=} (\Omega_x P)(\Omega_x P)$

$= \Omega_x (P \Omega_x) P \overset{(i)}{=} \Omega_x P^2$

$$= \Omega_{\mathcal{X}} P \overset{(ii)}{=} \Omega_{\mathcal{X}}$$

Explanation of intermediate normalization:

$$\langle \Phi_0 | \Psi_{\mathcal{X}} \rangle = \langle \Phi_0 | \Omega_{\mathcal{X}} | \Phi_0 \rangle$$

$$= \langle \Phi_0 | \underbrace{P \Omega_{\mathcal{X}}}_{P \;\leftarrow\; \text{property (i)}} | \Phi_0 \rangle$$

$$= \langle \Phi_0 | P | \Phi_0 \rangle$$

$$= \langle \Phi_0 | \Phi_0 \rangle = 1.$$

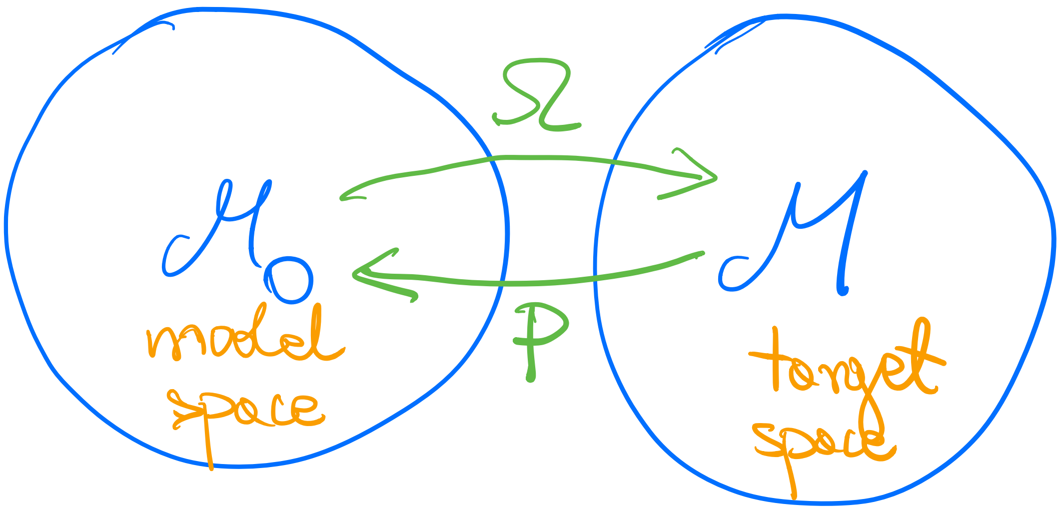
Ω (or $\Omega_{\mathcal{X}}$) is an example of the Bloch wave operator.

$$|\Phi_0\rangle \xrightarrow{\Omega} |\Psi_0\rangle$$

In general, we can introduce the zeroth-order space called the model space,

in our case M=1

$$\mathcal{M}_0 = \text{Span}\{|\Phi_p\rangle\}_{p=1}^{M}$$



$$\mathcal{M} = \text{Span}\{|\Psi_\mu\rangle\}_{\mu=1}^{M}$$

$$\begin{cases} \Omega|_{\mathcal{M}_0} = (P_{\mathcal{M}})^{-1} \\ \Omega|_{\mathcal{M}_0^\perp} = 0 \end{cases}$$

Going back to RSPT:

$$K = K_0 + W,$$

$$K_0 |\Phi_n\rangle = \varkappa_n |\Phi_n\rangle, \quad n = 0, 1, \ldots$$

$$|\Psi_\varkappa\rangle = \Omega_\varkappa |\Phi_0\rangle$$

$$\Omega_\varkappa = P + Q_\varkappa K P \quad \checkmark$$

$$f(\varkappa) = \langle \Phi_0 | K \Omega_\varkappa | \Phi_0 \rangle \quad \checkmark$$

$$\Omega_\varkappa = P + Q_\varkappa (K_0 + W) P$$

$$= P + Q_\varkappa K_0 P + Q_\varkappa W P$$

$$Q_x K_0 P = Q_x \underbrace{K_0 |\Phi_0\rangle}_{x_0|\Phi_0\rangle}\langle\Phi_0|$$

$$= x_0 Q_x |\Phi_0\rangle\langle\Phi_0|$$

$$= x_0 Q_x P \overset{(2)}{=} 0$$

$$\Omega_x = P + Q_x W P$$

$$f(x) = \langle\Phi_0|(K_0 + W)\Omega_x|\Phi_0\rangle$$

$$= \underbrace{\langle\Phi_0|K_0|\Phi_0\rangle}_{x_0\langle\Phi_0|\Phi_0\rangle = x_0} + \langle\Phi_0|W\Omega_x|\Phi_0\rangle$$

$$= x_0 + \langle\Phi_0|\tau_x|\Phi_0\rangle$$

$$\tau_x = PW\Omega_x = PWP + PWQ_xWP$$

In RSPT:

$$|\Psi_\varkappa\rangle = \Omega_\varkappa |\Phi_0\rangle$$

$$f(\varkappa) = \varkappa_0 + \langle\Phi_0|\tau_\varkappa|\Phi_0\rangle,$$

where

$$\Omega_\varkappa = P + Q_\varkappa W P$$

$$\tau_\varkappa = P W \Omega_\varkappa$$

$$= PWP + PWQ_\varkappa WP$$

In particular, for $\varkappa = k_0$,

$$|\Psi_0\rangle = \Omega|\Phi_0\rangle$$

$f(k_0)=k_0$

$$k_0 = \varkappa_0 + \langle\Phi_0|\tau|\Phi_0\rangle,$$

where

$$\Omega = P + Q_{k_0} W P,$$

$$\tau = PW\Omega$$
$$= PWP + PWR_{K_0}WP.$$

Furthermore,

$$\Omega P = \Omega, \quad P\Omega = P,$$
$$\Omega^2 = \Omega.$$

We introduce the reduced resolvent of K_0 at $\varkappa = \varkappa_0$,

$$R^{(0)} = R_{\varkappa=\varkappa_0}(K_0)$$
$$= Q\left[\alpha P + Q(\varkappa_0 - K_0)Q\right]^{-1} Q$$

$R^{(0)}$ – unperturbed reduced resolvent

exists since $Q(\varkappa_0 - K_0)Q$ can be inverted in $\mathcal{H}_Q$ ($\varkappa_0$ is a nondegenerate eigenvalue of K_0)

$$(K_0|\Phi_n\rangle = \varkappa_n|\Phi_n\rangle)$$

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} (\varkappa_0-\varkappa_1)^{-1} & & 0 \\ & (\varkappa_0-\varkappa_2)^{-1} & \\ 0 & & \ddots \end{matrix} \end{array}\right) \leftarrow (\varkappa_0-\varkappa_n)^{-1},\ n \geq 1, \text{ on the diagonal}$$

We will now represent $R_\varkappa(K)$ in terms of $R^{(0)}$ and W.

We will use

$$(A-B)^{-1} = A^{-1} + A^{-1}B(A-B)^{-1}$$

$$R_\varkappa(K) = Q[\alpha P + Q(\varkappa - K)Q]^{-1}Q$$

$$= Q[\alpha P + Q(\varkappa_0 - K_0 + \varkappa - \varkappa_0 - W)Q]^{-1}Q$$

$$= Q[\alpha P + Q(\varkappa_0 - K_0)Q - QW'_\varkappa Q]^{-1}Q$$

$$W'_{x} = W - (x - x_0)$$

$\underline{\qquad} \longrightarrow A$

$\underline{\qquad} \longrightarrow B$

$$= Q\left[\underline{\alpha P + Q(x_0 - K_0)Q}\right]^{-1} Q$$

$$+ Q\left[\underline{\alpha P + Q(x_0 - K_0)Q}\right]^{-1} \underline{Q W'_{x} Q}$$

$$\times \left[\alpha P + Q(x - K)Q\right]^{-1} Q$$

$$= R^{(0)} + R^{(0)} W'_{x} R_{x}$$

$$R_{x} = R^{(0)} + R^{(0)} W'_{x} R_{x}$$

$$W'_{x} = W - (x - x_0)$$

$$
\begin{aligned}
R_{\mathfrak{X}} &= R^{(0)} + R^{(0)} W_{\mathfrak{X}}' \left(R^{(0)} + R^{(0)} W_{\mathfrak{X}}' R_{\mathfrak{X}} \right) \\
&= R^{(0)} + R^{(0)} W_{\mathfrak{X}}' R^{(0)} + R^{(0)} W_{\mathfrak{X}}' R^{(0)} W_{\mathfrak{X}}' R_{\mathfrak{X}} \\
&= R^{(0)} + \left(R^{(0)} W_{\mathfrak{X}}' \right) R^{(0)} + \left(R^{(0)} W_{\mathfrak{X}}' \right)^2 \left(R^{(0)} + R^{(0)} W_{\mathfrak{X}}' R_{\mathfrak{X}} \right) \\
&= R^{(0)} + \left(R^{(0)} W_{\mathfrak{X}}' \right) R^{(0)} + \left(R^{(0)} W_{\mathfrak{X}}' \right)^2 R^{(0)}
\end{aligned}
$$

$$+\left(R^{(0)}W_x'\right)^3 R_x = \cdots =$$

$$= R^{(0)} + \sum_{n=1}^{\infty}\left(R^{(0)}W_x'\right)^n R^{(0)}$$

$$= \sum_{n=0}^{\infty}\left(R^{(0)}W_x'\right)^n R^{(0)}$$

$$R_x(K) = \sum_{n=0}^{\infty}\left(R^{(0)}W_x'\right)^n R^{(0)}$$

$$W_x' = W - (x - x_0)$$

$$|\Psi_\varkappa\rangle = \Omega_\varkappa |\Phi_0\rangle$$

$$\Omega_\varkappa = P + Q_\varkappa W P$$

$$= P + \sum_{n=0}^{\infty} \left(R^{(0)} W'_\varkappa\right)^n R^{(0)} W P$$

$$f(\varkappa) = \varkappa_0 + \langle \Phi_0 | \tau_\varkappa | \Phi_0 \rangle$$

$$\tau_\varkappa = P W \Omega_\varkappa$$

$$= PWP + \sum_{n=0}^{\infty} PW \left(R^{(0)} W'_\varkappa\right)^n R^{(0)} W P$$

$$W'_\varkappa = W - (\varkappa - \varkappa_0)$$

What interests us is $\varkappa = k_0$.
We obtain

$$|\Psi_0\rangle = \Omega|\Phi_0\rangle$$

$$\Omega = P + Q_{k_0} W P$$

$$= P + \sum_{n=0}^{\infty} (R^{(0)} W')^n R^{(0)} W P$$

$$k_0 = \varepsilon_0 + \langle\Phi_0|\tau|\Phi_0\rangle$$

$$\tau = P W \Omega$$

$$= PWP + \sum_{n=0}^{\infty} PW(R^{(0)} W')^n R^{(0)} W P$$

$$W' = W - (k_0 - \varepsilon_0)$$

$$k_0 = \varepsilon_0 + \sum_{m=1}^{\infty} k_0^{(m)}$$

$$W' = W - \sum_{m=1}^{\infty} k_0^{(m)}$$

$$|\Psi_0\rangle = |\Phi_0\rangle + \sum_{n=1}^{\infty} |\Psi_0^{(n)}\rangle$$

$$|\Psi_0\rangle = \underbrace{|\Phi_0\rangle}_{\text{0th order}}$$

$$+ \underbrace{\sum_{n=0}^{\infty} (R^{(0)}W')^n R^{(0)} W |\Phi_0\rangle}_{\text{at least 1st order}}$$

$$k_0 = \underbrace{\varepsilon_0}_{\text{0th order}} + \underbrace{\langle\Phi_0|W|\Phi_0\rangle}_{\text{1st order}}$$

$$+ \underbrace{\sum_{n=0}^{\infty} \langle\Phi_0|W(R^{(0)}W')^n R^{(0)} W|\Phi_0\rangle}_{\text{at least 2nd order}}$$

$$W' = W - (\overbrace{k_0 - \varepsilon_0}^{\sum_{m=1}^{\infty} k_0^{(m)}})$$

at least 1st order

W in W' results in the principal term in each order.
$(k_0 - \varkappa_0)$ contributions in W' result in renormalization terms.