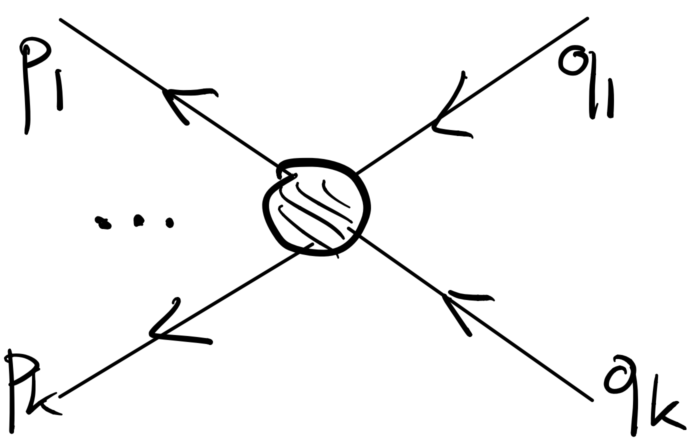
HUGENHOLTZ AND BRANDOW DIAGRAMS

$$\hat{O}_k = \left(\frac{1}{k!}\right)^2 \sum_{\substack{p_1\dots p_k \\ q_1\dots q_k}} \langle p_1\dots p_k|\hat{o}_k|q_1\dots q_k\rangle_A$$

$$\times\ X^{+}_{p_1}\dots X^{+}_{p_k} X_{q_k}\dots X_{q_1}$$

$$\left(\text{or } N[X^{+}_{p_1}\dots X^{+}_{p_k} X_{q_k}\dots X_{q_1}] \ \text{ or } \ (N[X^{+}_{p_1}X_{q_1}\dots X^{+}_{p_k}X_{q_k}])\right)$$



$$g_{\hat{O}_k} = (k!)^2$$

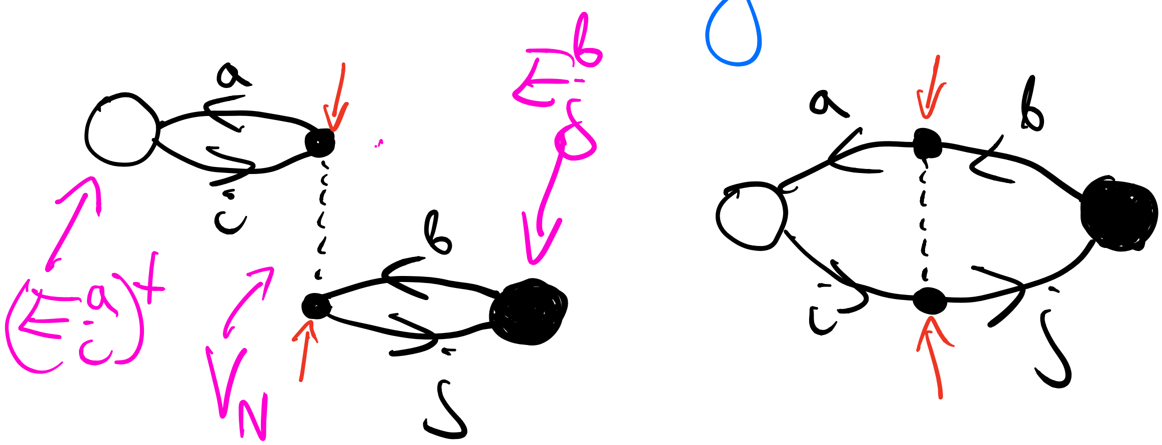
$$W^{(H)}_{\hat{O}_k} = \left(\frac{1}{k!}\right)^2$$

Goldstone $\xrightarrow{\text{homomorphism}}$ Hugenholtz

Examples:

$$-\langle \Phi_i^a | V_N | \Phi_j^b \rangle$$

Two Goldstone diagrams:



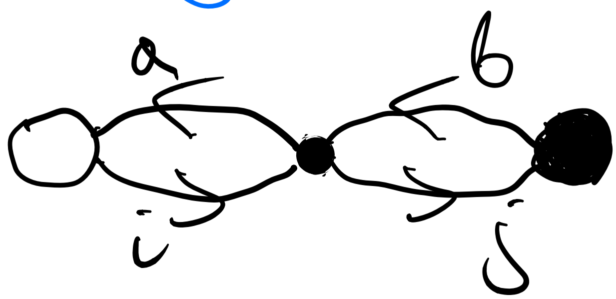
$$+\langle aj|\hat{o}|ib\rangle \qquad -\langle aj|\hat{o}|bi\rangle$$

Result: $\langle aj|\hat{o}|ib\rangle_A$ W

Let us collapse the simple vertices of V_N:

$$\Downarrow$$

Hugenholtz diagram



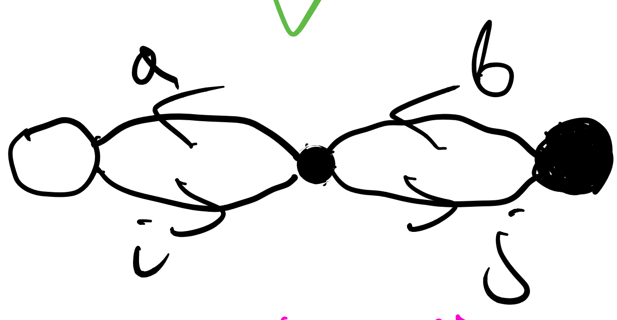
✓

pq, rs – free
a, b, i, j – fixed

Verification (directly from Hugenholtz vertices):

$\Downarrow$

$w^{(H)}=1$



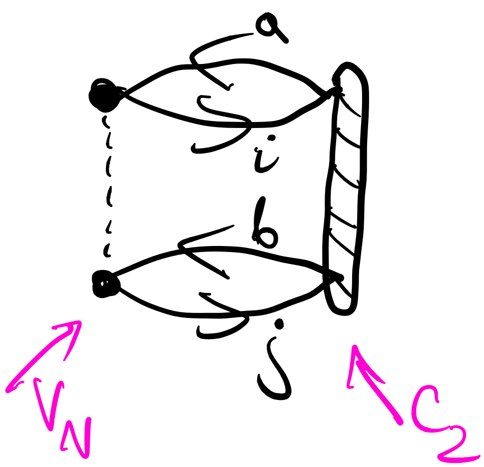
✓

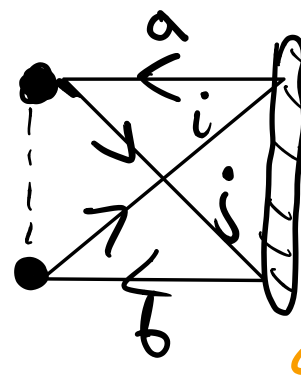
$\pm \langle aj|\hat{v}|ib\rangle_A$ ✓

(or $\pm \langle aj|\hat{v}|bi\rangle_A$)

$$-\langle \Phi | V_N C_2 | \Phi \rangle$$

Two Goldstone diagrams:





a,b,i,j free

$$+\frac{1}{2}\sum_{abij}\langle ij|\hat{v}|ab\rangle \times \langle ab|\hat{c}_2|ij\rangle \qquad -\frac{1}{2}\sum_{abij}\langle ij|\hat{v}|ba\rangle \times \langle ab|\hat{c}_2|ij\rangle$$

Result:

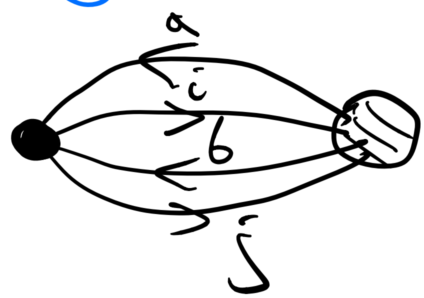
$$\frac{1}{2}\sum_{abij}\langle ij|\hat{v}|ab\rangle_A \langle ab|\hat{c}_2|ij\rangle$$

$$\left(\text{or } \frac{1}{2}\sum_{abij}\langle ij|\hat{v}|ab\rangle\langle ab|\hat{c}_2|ij\rangle_A\right)$$

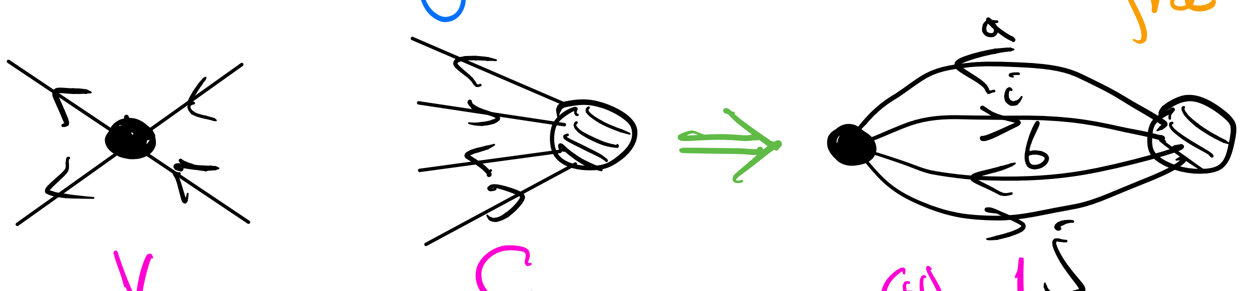
Let us collapse the simple vertices of V_N:

$$\Downarrow$$

Hugenholtz diagram



Verification (directly from Hugenholtz vertices): a,b,i,j free



V_N $\quad$ C_2 $\quad$ $W^{(H)} = \frac{1}{4}$

$$\pm \frac{1}{4} \sum_{abij} \langle ij|\hat{v}|ab\rangle_A \langle ab|\hat{C}_2|ij\rangle_A$$

This is a correct expression up to a sign:

$$\pm \frac{1}{4} \sum_{abij} \langle ij|\hat{v}|ab\rangle_A \langle ab|\hat{C}_2|ij\rangle_A$$

$$= \pm \frac{1}{4} \left(\sum_{abij} \langle ij|\hat{v}|ab\rangle_A \langle ab|\hat{C}_2|ij\rangle - \sum_{abij} \langle ij|\hat{v}|ab\rangle_A \langle ab|\hat{C}_2|ji\rangle \right)$$

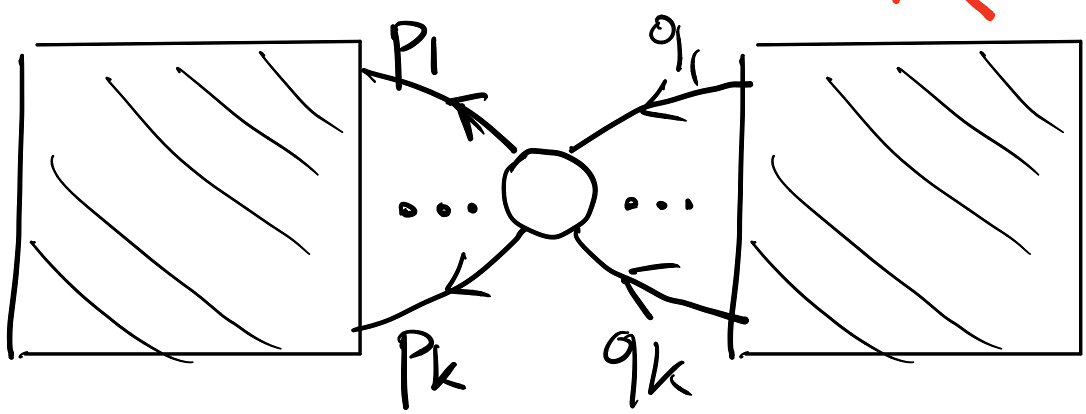
$$- \sum_{abji} \langle ji|\hat{v}|ab\rangle_A \langle ab|\hat{C}_2|ij\rangle$$

$$+ \sum_{abij} \langle ij|\hat{v}|ab\rangle_A \langle ab|\hat{C}_2|ij\rangle$$

$$= \pm \frac{1}{2} \sum_{abij} \langle ij|\hat{v}|ab\rangle_A \langle ab|\hat{C}_2|ij\rangle$$

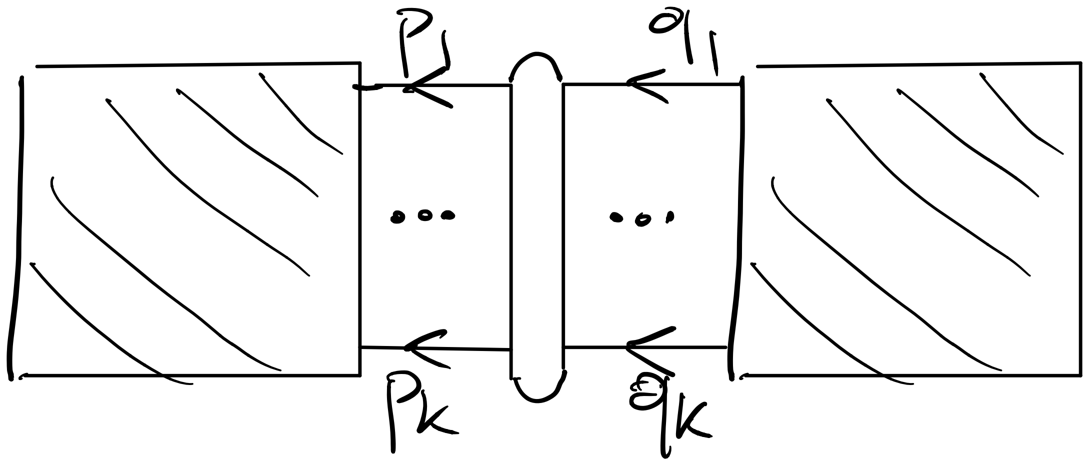
The correspondence between Hugenholtz and Goldstone d-ms allows us to determine the sign.

R



$$\pm w_R^{(H)} \sum_{\substack{\dots p_1 \dots p_k \dots \\ \dots q_1 \dots q_k \dots}} \dots \langle p_1 \dots p_k | \hat{o}_k | q_1 \dots q_k \rangle_A \dots$$

Among Goldstone d-ms correspon-ding to Hugenholtz d-m R, we have



Reading this Goldstone gives

$$w^{(G)} (-1)^{l_R^{(G)} + h_R^{(G)}}$$

$$\times \sum_{\substack{\ldots p_1 \ldots p_k \ldots \\ \ldots q_1 \ldots q_k \ldots}} \ldots \langle p_1 \ldots p_k | \hat{O}_k | q_1 \ldots q_k \rangle \ldots$$

Let us look at the Hugenholtz d-m R again:

$$\pm w_R^{(H)} \sum_{\substack{\cdots p_1 \cdots p_k \cdots \\ \cdots q_1 \cdots q_k \cdots}} \cdots \langle p_1 \cdots p_k | \hat{o}_k | q_1 \cdots q_k \rangle_A \cdots$$

$$= \pm w_R^{(H)} \sum_{\substack{\cdots p_1 \cdots p_k \cdots \\ \cdots q_1 \cdots q_k \cdots}} \cdots \Big(\underline{\underline{\langle p_1 \cdots p_k | \hat{o}_k | q_1 \cdots q_k \rangle}}$$

$$- \langle p_1 p_2 \cdots p_k | \hat{o}_k | q_2 q_1 \cdots q_k \rangle + \cdots \Big)$$

$$(-1)^{l_R^{(G)} + h_R^{(G)}}$$

Because of this procedure, we call the Goldstone representative of a given Hugenholtz d-m a BRANBOW d-m.

Procedure used in Hugenholtz diagrammatics is as follows:

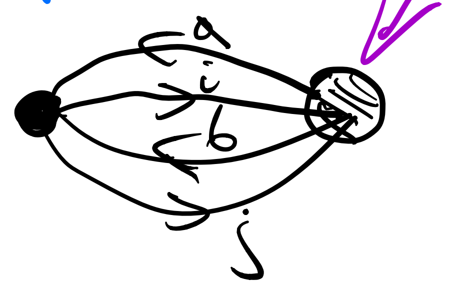
1. Construct the resulting Hugenholtz d-ms.
2. Determine Hugenholtz weights.
3. Represent each Hugenholtz d-m by its Brandow version.
4. Assign Hugenholtz scalar factors (antisymmetrized matrix elements) to vertices of Brandow digrams.

5. Determine the sign from each Brandow d-m as

$$(-1)^{l_R^{(B)}+h_R^{(B)}}$$

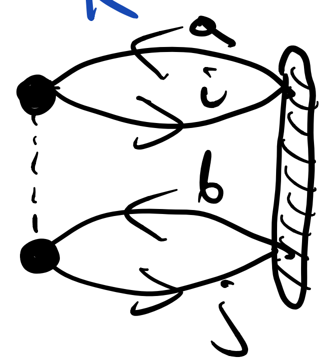
Example:

$$\langle \Phi | V_N S_2 | \Phi \rangle$$

1. 

a, b, i, j -free Hugenholtz (H)

2. $$w_R^{(H)} = \frac{1}{4}.$$

3. 

Brandow (B)

4. $$d^{(B)}_{\chi_R'} = \langle ij|\hat{o}|ab\rangle_A \langle ab|\hat{c}_2|ij\rangle_A$$

5. $$S_R^{(B)} = (-1)^{l_R^{(B)} + h_R^{(B)}}$$
$$= (-1)^{2+2} = +1.$$

Resulting expression:

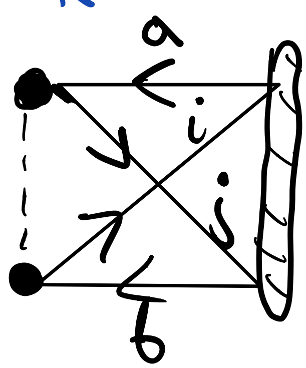
$$S_R^{(B)} \; W_R^{(H)} \sum_{X_R'} d_{X_R'}^{(B)}$$

$$= +\frac{1}{4} \sum_{abij} \langle ij|\hat{v}|ab\rangle_A \langle ab|\hat{c}_2|ij\rangle_A$$

What if we choose a different Goldstone or Brandow?

1. Done above.

2. $w_R^{(H)} = \frac{1}{4}$.

3.  $\leftarrow$ Brandow

4. $$d^{(B)}_{\chi_R'} = \langle ij | \hat{v} | ba \rangle_A \times \langle ab | \hat{c}_2 | ij \rangle_A .$$

5. $s_R^{(B)} = (-1)^{1+2} = -1$.

Resulting expression:

$$s_R^{(B)} \, w_R^{(H)} \sum_{\chi_R'} d^{(B)}_{\chi_R'}$$

$$= -\frac{1}{4} \sum_{abij} \langle ij | \hat{v} | ba \rangle_A \langle ab | \hat{c}_2 | ij \rangle_A$$

$$= +\frac{1}{4} \sum_{abij} \langle ij | \hat{v} | ab \rangle_A \langle ab | \hat{c}_2 | ij \rangle_A$$

(same as before)

Hugenholtz d-ms can also be used as an intermediate step toward determining all Goldstone d-ms.

1. Start by constructing NONORIENTED resulting Hugenholtz diagrams.

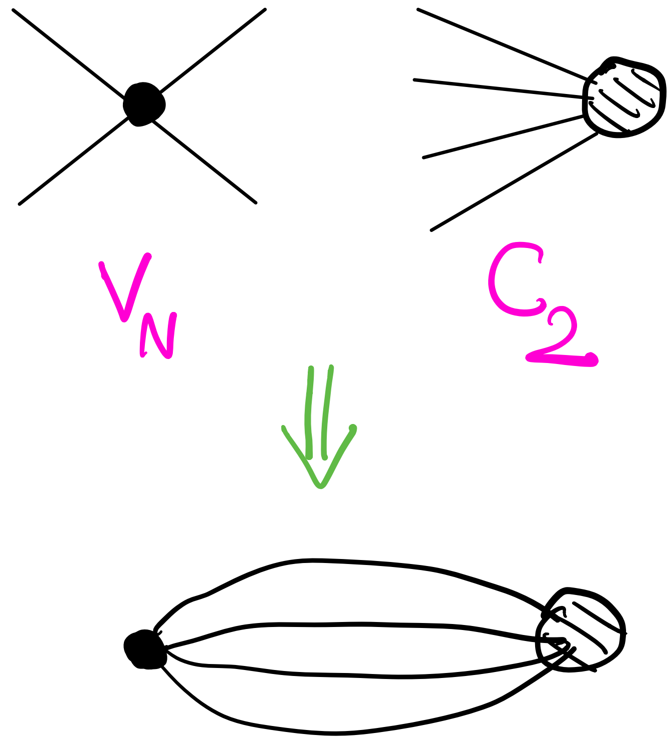
2. Insert arrows into all Fermion lines and label the Fermion lines accordingly (according to the left-right orientation).

3. Expand each Hugenholtz vertex to produce first one Goldstone d-m per each Hugenholtz d-m and generate the remaining Goldstone d-ms by "exchanging" OUTGOING OR INCOMING (but NOT BOTH) Fermion lines in the initial Goldstone representatives.

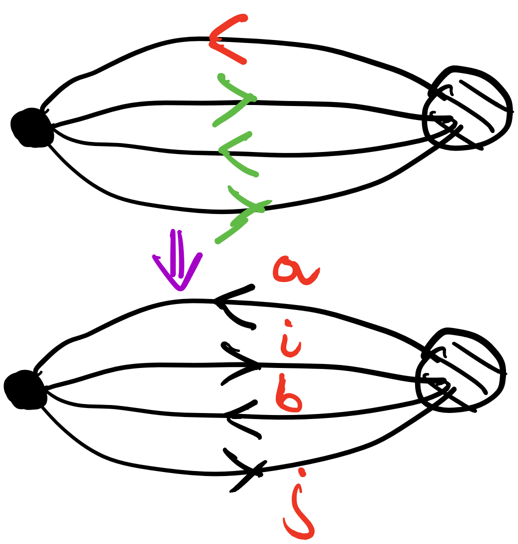
Example:

$$\langle \Phi | V_N C_2 | \Phi \rangle$$

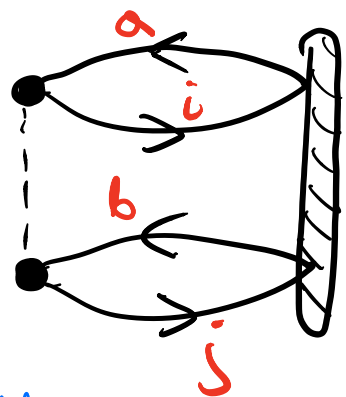
1.



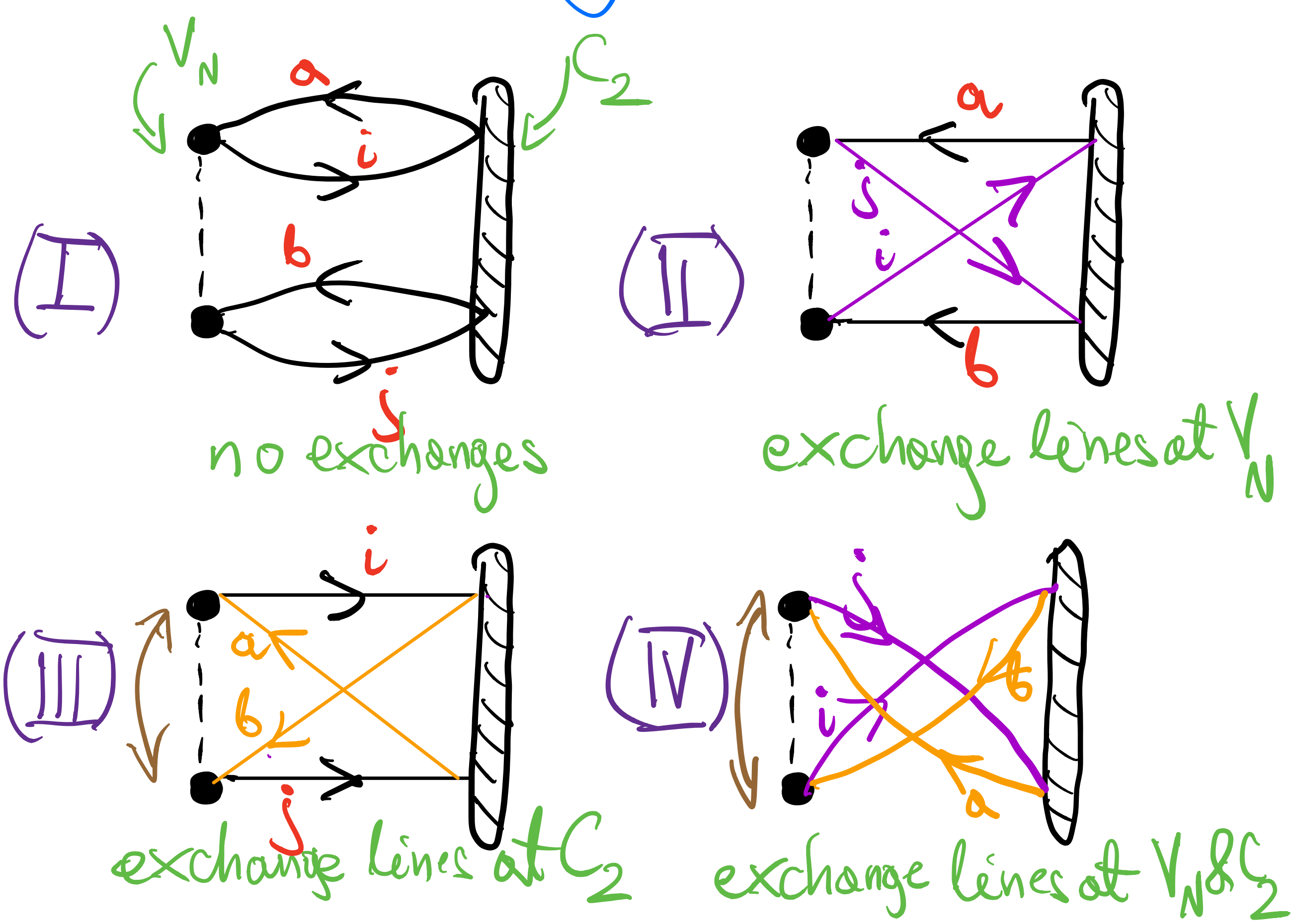
2.



3.

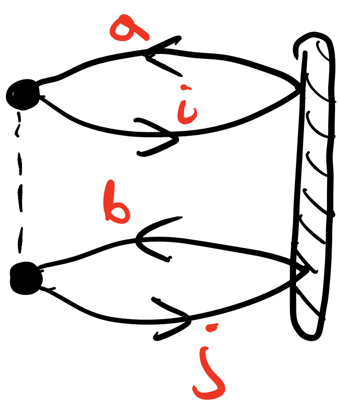


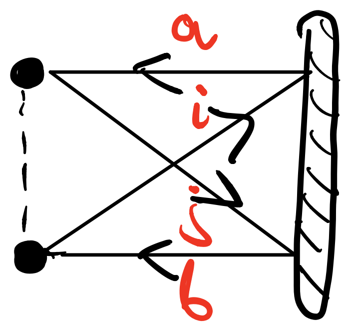
We will produce the remaining Goldstone d-ms by line exchanges at V_N and C_2.
We will do it using OUTGOING lines.



$$(\mathrm{I}) = (\mathrm{IV})$$
$$(\mathrm{II}) = (\mathrm{III})$$

Final answer:





Next topic:

SPIN INDEPENDENT OPERATORS

(watch video 24)

THE TOPIC AFTER THIS:

Independent Particle Models (IPMs)

$$|\Psi\rangle \simeq |\Phi\rangle$$

single Slater determinant

We will study IPMs using THOULESS' THEOREM

Thouless theorem:

$$|\Phi\rangle,\ |\tilde{\Phi}\rangle,\ \langle\Phi|\tilde{\Phi}\rangle \neq 0,$$

$$|\tilde{\Phi}\rangle \sim e^{\hat{C}_1}|\Phi\rangle$$

More precisely, if $\langle\Phi|\tilde{\Phi}\rangle = 1$

then

$$|\tilde{\Phi}\rangle = e^{\hat{C}_1}|\Phi\rangle,$$

where

$$\hat{C}_1 = \sum_{i,a} c_{ia} E_i^a,$$

with

$$E_i^a = X_a^+ X_i .$$

Proof:

Consider

$$|\Phi\rangle = |\{1 2 \ldots N\}\rangle$$
$$= \prod_{i=1}^{N} X_i^{\dagger} |0\rangle ,$$

$$|\tilde{\Phi}\rangle = |\{\tilde{1} \tilde{2} \ldots \tilde{N}\}\rangle$$
$$= \prod_{i=1}^{N} \tilde{X}_i^{\dagger} |0\rangle .$$

Old basis: $\{|p\rangle\}$ (to construct $|\Phi\rangle$)

New basis: $\{|\tilde{p}\rangle\}$ (to construct $|\tilde{\Phi}\rangle$)

$$|\tilde{p}\rangle = \sum_{p} \alpha_{\tilde{p}p} |p\rangle$$

$$\tilde{X}^{\dagger}_{\tilde{p}} |0\rangle = \sum_{p} \alpha_{\tilde{p}p} X^{\dagger}_{p} |0\rangle$$

$$\tilde{X}^{\dagger}_{\tilde{p}} = \sum_{p} \alpha_{\tilde{p}p} X^{\dagger}_{p}$$

In particular,

$$\tilde{X}^{\dagger}_{i} = \sum_{p} \alpha_{ip} X^{\dagger}_{p}, \qquad i = 1, \ldots, N.$$

$$\tilde{X}^{\dagger}_{i} = \sum_{j=1}^{N} \underbrace{f_{ij}}_{} X^{\dagger}_{j} + \sum_{b=N+1}^{(\infty)} g_{ib} X^{\dagger}_{b}$$

$$f_{ij} = \alpha_{ij}, \qquad g_{ib} = \alpha_{ib}$$

What is the consequence of the assumption $\langle \Phi | \tilde{\Phi} \rangle = 1$ on coefficients f_{ij}?

$$\langle \Phi | \tilde{\Phi} \rangle = \langle 0 | X_N \cdots X_1 \, \tilde{X}_1^\dagger \cdots \tilde{X}_N^\dagger | 0 \rangle$$

$$= \langle 0 | X_N \cdots X_1 \underbrace{\left(\sum_{j_1=1}^{N} f_{1 j_1} X_{j_1}^\dagger + \sum_{b_1=N+1}^{(\infty)} g_{1 b_1} X_{b_1}^\dagger \right)}_{\tilde{X}_1^\dagger}$$

$$\times \cdots \times \underbrace{\left(\sum_{j_N=1}^{N} f_{N j_N} X_{j_N}^\dagger + \sum_{b_N=N+1}^{(\infty)} g_{N b_N} X_{b_N}^\dagger \right)}_{\tilde{X}_N^\dagger} | 0 \rangle$$

$$= \sum_{j_1, \ldots, j_N = 1}^{N} f_{1 j_1} \cdots f_{N j_N} \underline{\langle 0 | X_N \cdots X_1 X_{j_1}^\dagger \cdots X_{j_N}^\dagger | 0 \rangle}$$

$$+\sum_{b_1=N+1}^{\infty}\sum_{j_2,\dots,j_N=1}^{N} g_{1b_1} f_{2j_2}\cdots f_{Nj_N}$$

$$\times \langle 0| X_N\cdots X_1 X_{b_1}^\dagger X_{j_2}^\dagger\cdots X_{j_N}^\dagger|0\rangle$$

$= 0$, $\underbrace{X_j X_b^\dagger}=0$

$$+\cdots =$$

$$=\sum_{\substack{j_1,\dots,j_N=1\\(j_1\neq\dots\neq j_N)}}^{N} f_{1j_1}\cdots f_{Nj_N}$$

$$\times \langle 0| X_N\cdots X_1 X_{j_1}^\dagger\cdots X_{j_N}^\dagger|0\rangle$$

(f.c.)

$$=\sum_{P\in S_N}(-1)^P f_{1j_1}\cdots f_{Nj_N}$$

$$\times \langle 0| X_N\cdots X_1 X_1^\dagger\cdots X_N^\dagger|0\rangle$$

$\dots$

$$P = \begin{pmatrix} 1 & \dots & N \\ j_1 & \dots & j_N \end{pmatrix}$$

$$\langle \Phi | \Phi \rangle = \sum_{P \in S_N} (-1)^P f_{1 j_1} \cdots f_{N j_N}$$

$$= \det \| f_{ij} \|_{i,j=1}^{N}$$

$$\langle \Phi | \Phi \rangle = 1$$

$$\Downarrow$$

$$\det \underbrace{\| f_{ij} \|_{i,j=1}^{N}}_{\underset{\sim}{f}} = 1$$

$\underset{\sim}{f}$ ← bold f (matrix $\|f\|_{i,j=1}^{N}$)

$$\det \underset{\sim}{f} = 1.$$

$$\underset{\sim}{f}^{-1} \text{ exists.}$$

We now transform $|\tilde{\Phi}\rangle$ to $|\tilde{\tilde{\Phi}}\rangle$,

$$|\tilde{\tilde{\Phi}}\rangle = |\{\tilde{\tilde{1}}\,\tilde{\tilde{2}} \ldots \tilde{\tilde{N}}\}\rangle$$

$$= \prod_{i=1}^{N} \tilde{\tilde{X}}_i^{\dagger} |0\rangle,$$

where the basis used to define $|\tilde{1}\rangle, \ldots, |\tilde{N}\rangle$ is defined as

$$|\tilde{i}\rangle = \sum_{j=1}^{N} (\underset{\sim}{f}^{-1})_{ij} |j\rangle.$$

$$\tilde{X}_i^\dagger = \sum_{j=1}^{N} (f^{-1})_{ij}\, \tilde{X}_j^\dagger$$

Let us determine $|\tilde{\Phi}\rangle$:

$$|\tilde{\Phi}\rangle = \tilde{X}_1^\dagger \cdots \tilde{X}_N^\dagger |0\rangle$$

$$= \sum_{j_1,\dots,j_N=1}^{N} (f^{-1})_{1j_1} \cdots (f^{-1})_{Nj_N} \times \tilde{X}_{j_1}^\dagger \cdots \tilde{X}_{j_N}^\dagger |0\rangle$$

$j_1 \neq \dots \neq j_N$

$(\tilde{X}_j^\dagger)^2 = 0$

and

$\{\tilde{X}_j^\dagger, \tilde{X}_j^\dagger\} = 0$

$$= \sum_{1 \leq j_1 \neq \dots \neq j_N}^{N} (f^{-1})_{1 j_1} \cdots (f^{-1})_{N j_N}$$

$$\times \tilde{X}^{+}_{j_1} \cdots \tilde{X}^{+}_{j_N} |0\rangle$$

$$= \sum_{P \in S_N} (-1)^P (f^{-1})_{1 j_1} \cdots (f^{-1})_{N j_N} \times \underbrace{\tilde{X}^{+}_{1} \cdots \tilde{X}^{+}_{N} |0\rangle}_{|\tilde{\Phi}\rangle}$$

$$P = \begin{pmatrix} 1 & \dots & N \\ j_1 & \dots & j_N \end{pmatrix}$$

$$|\tilde{\Phi}\rangle = \det f^{-1} |\Phi\rangle$$

$$1$$

$$|\tilde{\Phi}\rangle = |\Phi\rangle \, .$$

$$\tilde{X}_i^\dagger = \sum_{j=1}^{N} (f^{-1})_{ij} \, \tilde{X}_j^\dagger$$

$$= \sum_{j=1}^{N} (f^{-1})_{ij} \left(\sum_{k=1}^{N} f_{jk} X_k^\dagger + \sum_{b=N+1}^{(\infty)} g_{jb} X_b^\dagger \right)$$

$$\tilde{X}_i^\dagger = \sum_{j,k=1}^{N} (f^{-1})_{ij} \, f_{jk} X_k^\dagger + \sum_{j=1}^{N} \sum_{b=N+1}^{(\infty)} (f^{-1})_{ij} \, g_{jb} X_b^\dagger$$

$$\tilde{\tilde{X}}_i^{\dagger} = \sum_{k=1}^{N} \underbrace{\left(\underset{\sim}{f}^{-1} \bullet \underset{\sim}{f}\right)_{ik}}_{=\delta_{ik}} X_k^{\dagger}$$

$$= \sum_{b=N+1}^{(\infty)} \left(\underset{\sim}{f}^{-1} \bullet \underset{\sim}{g}\right)_{ib} X_b^{\dagger}$$

$$N\ \overset{N}{\boxed{\underset{\sim}{f}^{-1}}} \bullet N\ \overset{(\infty)}{\boxed{\qquad\qquad \underset{\sim}{g} \qquad\qquad}}$$

$$= N\ \underset{(\infty)}{\boxed{\qquad\qquad \underset{\sim}{c} \qquad\qquad}}$$

$$c_{ia} = \left(\underset{\sim}{f}^{-1} \bullet \underset{\sim}{g}\right)_{ia}$$

$$= \sum_{j=1}^{N} \left(f^{-1}\right)_{ij} g_{ja}$$

$$\tilde{\tilde{X}}_i^\dagger = X_i^\dagger + \sum_{a=N+1}^{(\infty)} c_{ia} X_a^\dagger$$

The above were preparatory steps. Now the main part of the proof.

$$|\tilde{\Phi}\rangle = |\tilde{\tilde{\Phi}}\rangle = \prod_{i=1}^{N} \tilde{\tilde{X}}_i^\dagger |0\rangle$$

$$= \prod_{i=1}^{N} \left(X_i^\dagger + \sum_a c_{ia} X_a^\dagger \right) |0\rangle$$

$$= \left(X_1^\dagger + \sum_{a_1} c_{1a_1} X_{a_1}^\dagger\right)$$

$$\times \left(X_2^\dagger + \sum_{a_2} c_{2a_2} X_{a_2}^\dagger\right)$$

$$\cdots$$

$$\times \left(X_N^\dagger + \sum_{a_N} c_{Na_N} X_{a_N}^\dagger\right)|0\rangle$$

$$X_1 X_1^\dagger X_2 X_2^\dagger \cdots X_N X_N^\dagger$$

We used

$$X_i X_i^\dagger |0\rangle = (1 - X_i^\dagger X_i)|0\rangle$$

$$= |0\rangle .$$

and

$$[X_i X_i^\dagger, X_j^\dagger] = 0 , \quad i \neq j$$

$$[X_i X_i^\dagger, X_a^\dagger] = 0 .$$

$$|\tilde{\Phi}\rangle = |\tilde{\tilde{\Phi}}\rangle$$

$$= \Big(\underbrace{X_1^\dagger X_1 X_1^\dagger}_{X_1^\dagger} + \sum_{a_1} c_{1a_1} X_{a_1}^\dagger X_1 X_1^\dagger\Big)$$

$$\times \Big(\underbrace{X_2^\dagger X_2 X_2^\dagger}_{X_2^\dagger} + \sum_{a_2} c_{2a_2} X_{a_2}^\dagger X_2 X_2^\dagger\Big)$$

$$\dots$$

$$\times \Big(\underbrace{X_N^\dagger X_N X_N^\dagger}_{X_N^\dagger} + \sum_{a_N} c_{Na_N} X_{a_N}^\dagger X_N X_N^\dagger\Big)|0\rangle$$

$$X_i^\dagger X_i X_i^\dagger = X_i^\dagger (1 - X_i^\dagger X_i) = X_i^\dagger$$

$$|\tilde{\Phi}\rangle = |\tilde{\tilde{\Phi}}\rangle$$

$$= \left(X_1^\dagger + \sum_{a_1} c_{1a_1} X_{a_1}^\dagger X_1 X_1^\dagger \right)$$

$$\times \left(X_2^\dagger + \sum_{a_2} c_{2a_2} X_{a_2}^\dagger X_2 X_2^\dagger \right)$$

$$\cdots$$

$$\times \left(X_N^\dagger + \sum_{a_N} c_{Na_N} X_{a_N}^\dagger X_N X_N^\dagger \right) |0\rangle$$

$$|\tilde{\Phi}\rangle = |\tilde{\Phi}\rangle$$

$$= \left(1 + \sum_{a_1} c_{1a_1} X^\dagger_{a_1} X_1\right) X_1^\dagger$$

$$\times \left(1 + \sum_{a_2} c_{2a_2} X^\dagger_{a_2} X_2\right) X_2^\dagger$$

$$\cdots$$

$$\times \left(1 + \sum_{a_N} c_{Na_N} X^\dagger_{a_N} X_N\right) X_N^\dagger |0\rangle$$

$$|\tilde{\Phi}\rangle = \left(1 + \sum_{a_1} c_{1a_1} X_{a_1}^{\dagger} X_1\right)$$

$$\times \cdots \times \left(1 + \sum_{a_N} c_{Na_N} X_{a_N}^{\dagger} X_N\right)$$

$$\times \underbrace{X_1^{\dagger} \cdots X_N^{\dagger} |0\rangle}_{|\Phi\rangle}$$

$$E_i^a = X_a^{\dagger} X_i\,, \quad \tilde{E}_i = \sum_a c_{ia} E_i^a$$

$$= \sum_a c_{ia} X_a^{\dagger} X_i$$

$$|\tilde{\Phi}\rangle = (1+\tilde{E}_1)(1+\tilde{E}_2)\cdots(1+\tilde{E}_N)|\Phi\rangle$$

$$= \prod_{i=1}^{N} (1+\tilde{E}_i)|\Phi\rangle$$

$$|\tilde{\Phi}\rangle = \Big(1 + \sum_i \tilde{E}_i + \sum_{i_1<i_2} \tilde{E}_{i_1}\tilde{E}_{i_2} + \dots + \tilde{E}_1\tilde{E}_2\cdots\tilde{E}_N\Big)|\Phi\rangle$$

$$|\tilde{\Phi}\rangle = \Big(1 + \sum_{r=1}^{N} \underbrace{\sum_{i_1<\dots<i_r} \tilde{E}_{i_1}\cdots\tilde{E}_{i_r}\Big)|\Phi\rangle}$$

$$\Downarrow$$

$$\sum_{r=1}^{N} \frac{1}{r!} \sum_{i_1\neq\dots\neq i_r} \tilde{E}_{i_1}\cdots\tilde{E}_{i_r}$$

$[\tilde{E}_i, \tilde{E}_j] = 0$, since $[E_i^a, E_j^b] = 0$

$$|\tilde{\Phi}\rangle = \left(1 + \sum_{r=1}^{N} \frac{1}{r!} \sum_{i_1 \neq \dots \neq i_r} \tilde{E}_{i_1} \cdots \tilde{E}_{i_r}\right) |\Phi\rangle$$

$$\tilde{E}_i^2 = \left(\sum_a c_{ia} E_i^a\right)^2$$

$$= \left(\sum_a c_{ia} E_i^a\right)\left(\sum_b c_{ib} E_i^b\right)$$

$$= \sum_{a,b} c_{ia} c_{ib} \underbrace{E_i^a E_i^b} = 0$$

$$x_a^\dagger x_i x_b^\dagger x_i = -x_a^\dagger x_b^\dagger x_i^2$$

$$= 0$$

$$|\tilde{\Phi}\rangle = \Big(1 + \sum_{r=1}^{N} \frac{1}{r!} \underbrace{\sum_{i_1,\dots,i_r} \tilde{E}_{i_1} \cdots \tilde{E}_{i_r}}_{C_1^r}\Big)|\Phi\rangle$$

$$C_1 = \sum_i \tilde{E}_i = \sum_{i,a} c_{ia} E_i^a$$

$$= \sum_{i,a} c_{ia} X_a^{+} X_i$$

$$|\tilde{\Phi}\rangle = \Big(1 + \sum_{r=1}^{N} \frac{1}{r!} C_1^r\Big)|\Phi\rangle$$

$C_1^r = 0$ for $r > N$,

$$C_1^r = \sum_{i_1,\dots,i_r=1}^{N} \tilde{E}_{i_1} \cdots \tilde{E}_{i_r}.$$

If $r > N$, at least two indices

among $i_1, \ldots, i_r$ are identical.

But $[\tilde{E}_i, \tilde{E}_j] = 0$ and

$\tilde{E}_i^2 = 0$, so that $C_1^r = 0$.

extended the summation from 1 to N to 1 to ∞

$$|\tilde{\Phi}\rangle = \left(1 + \sum_{r=1}^{\infty} \frac{1}{r!} C_1^r\right) |\Phi\rangle$$

$$= e^{C_1} |\Phi\rangle, \text{ where}$$

$$C_1 = \sum_{i,a} c_{ia} X_a^{\dagger} X_i$$

(1p-1h excitation operator)

⋮