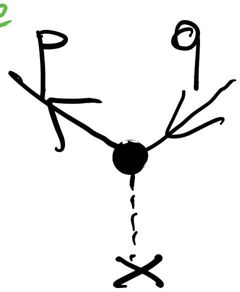
5

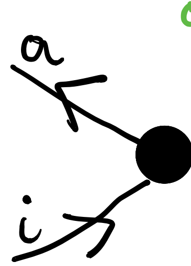
$$Z_N|\Phi_i^a\rangle = Z_N E_i^a|\Phi\rangle$$

$$E_i^a = N[X_a^\dagger X_i]$$

p,q - free



a,i - fixed

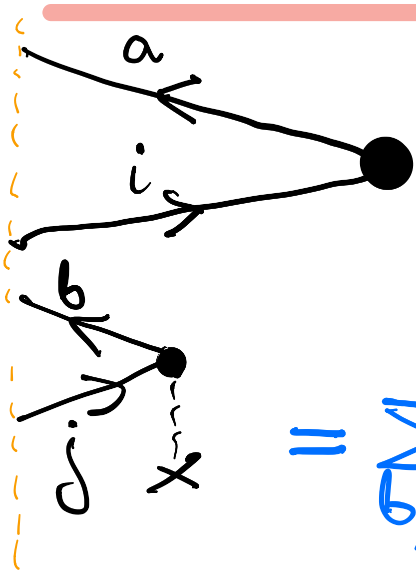


$$Z_N = \sum_{p,q} \langle p|\hat{z}|q\rangle \times N[X_p^\dagger X_q] \qquad E_i^a = N[X_a^\dagger X_i]$$

$|\Phi\rangle$

external lines in the resulting d-ms must extend to the left

No contractions



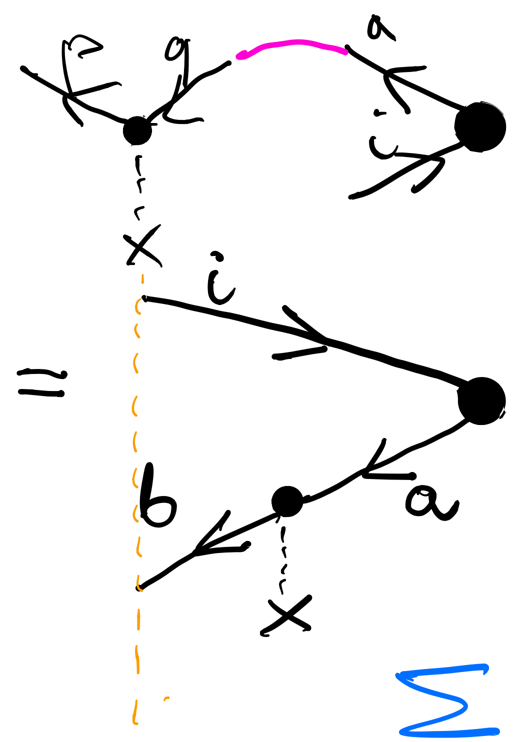
$$\sum_{p,q} \langle p|\hat{z}|q\rangle \times N[X_p^\dagger X_q X_a^\dagger X_i]|\Phi\rangle$$

occupied (j)

unoccupied (b)

$$= \sum_{b,j} \langle b|\hat{z}|j\rangle N[X_a^\dagger X_i X_b^\dagger X_j]|\Phi\rangle$$

One contraction



pq-free
a,i-fixed

b-free

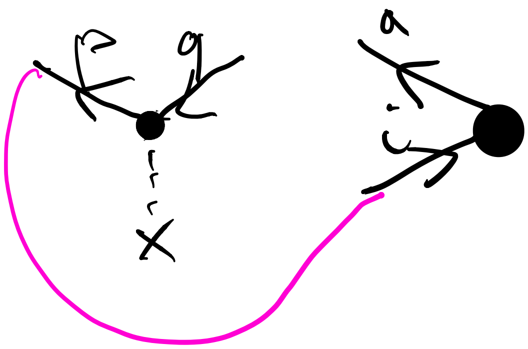
$$\sum_{p,q} \langle p|\hat{z}|q\rangle$$

$$\times\, N[X_p^\dagger \overline{X_q X_a^\dagger} X_i]|\Phi\rangle$$

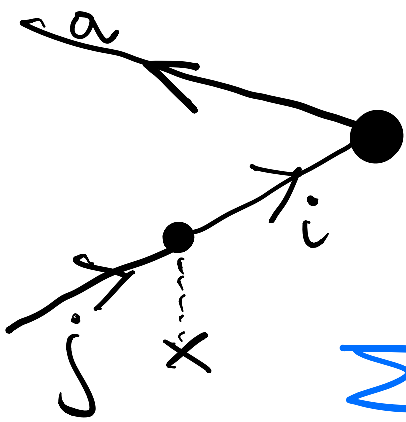
$$= \sum_{p,q} \langle p|\hat{z}|q\rangle\, \tilde{n}(q)\, \delta_{qa}\, N[X_p^\dagger X_i]|\Phi\rangle$$

unoccupied (b)

$$= \sum_{b} \langle b|\hat{z}|a\rangle\, N[X_b^\dagger X_i]|\Phi\rangle$$



p,q - free
a,i - fixed



j-free

$$\sum_{pq}\langle p|\hat{z}|q\rangle N[X_p^{\dagger}X_q X_a^{\dagger}X_i]|\Phi\rangle$$

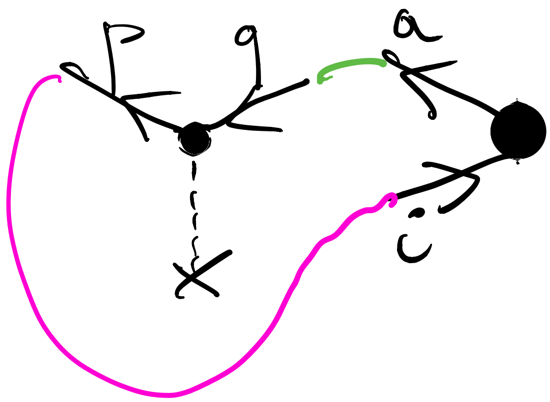
$$=\sum_{p,q}\langle p|\hat{z}|q\rangle\, x(p)\,\delta_{pi}\, N[X_q X_a^{\dagger}]|\Phi\rangle$$

occupied (j)

$$=\sum_{j}\langle i|\hat{z}|j\rangle N[X_j X_a^{\dagger}]|\Phi\rangle$$

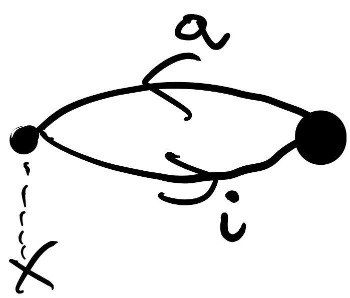
$$=-\sum_{j}\langle i|\hat{z}|j\rangle N[X_a^{\dagger}X_j]|\Phi\rangle$$

Two contractions



$|\Phi\rangle$ p,q - free

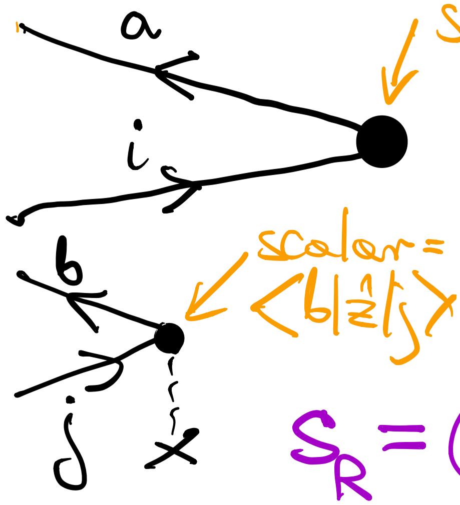
a,i - fixed



$$\sum_{p,q} \langle p|\hat{z}|q\rangle N[X_p^\dagger X_q X_a^\dagger X_i]|\Phi\rangle$$

$$= \langle i|\hat{z}|a\rangle \, |\Phi\rangle$$

(I)



$$\sum_{b,j} \langle b|\hat{z}|j\rangle \, |\Phi_{ij}^{ab}\rangle$$

$$s_R = (-1)^{l+h} = +1 \qquad l = 0, \quad h = 0$$

$$w_R = 1, \quad x_R' = \{\emptyset\}, \quad x_R'' = a,i,b,j$$

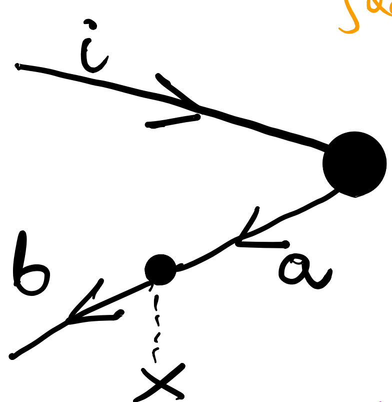
$$d^{(R)}_{x_R''} = \langle b|\hat{z}|j\rangle, \quad \hat{O}_{x_R''} = N[X_a^\dagger X_i X_b^\dagger X_j]$$

product of scalar factors

open path (a,i)

open path (b,j)

(II)



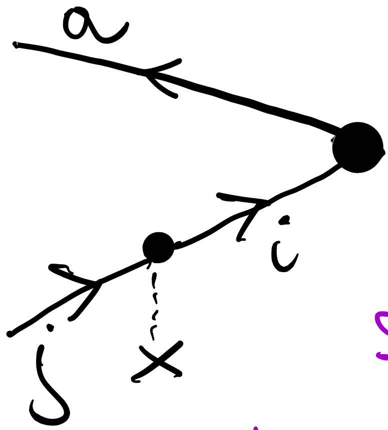
$$\sum_{b} \langle b|\hat{z}|a\rangle \, |\Phi_{i}^{b}\rangle$$

$$s_R = (-1)^{l+h} = (-1)^{0+0} = +1$$

$$w_R = 1, \quad x_R' = a, \quad x_R'' = b,j$$

$$d^{(R)}_{x_R', x_R''} = \langle b|\hat{z}|a\rangle, \quad \hat{O}_{x_R''} = N[X_b^\dagger X_i]$$

III



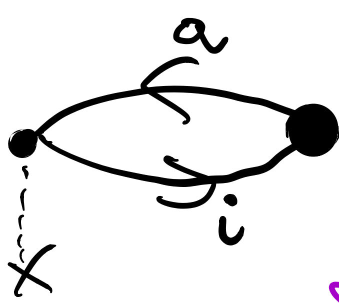
$$-\sum_j \langle i|\hat{z}|j\rangle\,|\Phi_j^a\rangle$$

$$s_R=(-1)^{l+h}=(-1)^{0+1}=-1$$

$$w_R=1,\ \chi_R'=i,\ \chi_R''=a,j$$

$$d^{(R)}_{\chi_R',\chi_R''}=\langle i|\hat{z}|j\rangle,\ \hat{O}_{\chi_R''}=N[X_a^\dagger X_j]$$

IV



$$\langle i|\hat{z}|a\rangle\,|\Phi\rangle$$

$$s_R=(-1)^{l+h}=(-1)^{1+1}=+1$$

$$w_R=1,\ \chi_R'=a,i,\ \chi_R''=\{\emptyset\}$$

$$d^{(R)}_{\chi_R'}=\langle i|\hat{z}|a\rangle,\ \hat{O}_{\chi_R''}=N[\emptyset]=1$$

$$Z_N |\Phi_i^a\rangle = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\overline{\mathrm{IV}})$$

$$Z_N |\Phi_i^a\rangle = Z_N E_i^a |\Phi\rangle = \sum_R K_R |\Phi\rangle$$

$$K_R = s_R w_R \sum_{\chi_R', \chi_R''} d^{(R)}_{\chi_R' \chi_R''} \hat{O}_{\chi_R''}$$

In general, $K_A \cdots K_Z = \sum_R K_R$.

We anticipate that weights have to be defined such that

the # of times we obtain d-m R

$$N_R \, w_A \cdots w_Z = w_R \, ,$$ ✓✓

so that $\sum_R$ involves only nonequivalent d-ms.

$$K_A \cdots K_Z = \sum_R K_R$$

nonequivalent resulting α-ms

$$\begin{cases} K_A = w_A \sum_{x_A''} \alpha^{(A)}_{x_A''} \hat{O}_{x_A''} \\ \cdots \\ K_Z = w_Z \sum_{x_Z''} \alpha^{(Z)}_{x_Z''} \hat{O}_{x_Z''} \end{cases}$$

$$K_R = s_R w_R \sum_{x_R', x_R''} \alpha^{(R)}_{x_R' x_R''} \hat{O}_{x_R''}$$

$$K_A \cdots K_Z = w_A \cdots w_Z$$
$$\times \sum_{x_A'', \ldots, x_Z''} d^{(A)}_{x_A''} \cdots d^{(Z)}_{x_Z''}$$
$$\times \hat{O}_{x_A''} \cdots \hat{O}_{x_Z''}$$

$$\overset{\text{(G)WT}}{=} \sum_R{}' K_R'$$ ✓

all terms originating from WT (or GWT)

$$K_R' = w_A \cdots w_Z \sum_{x_A'', \ldots, x_Z''} \underbrace{d^{(A)}_{x_A''} \cdots d^{(Z)}_{x_Z''}}_{\Rightarrow\, d^{(R)}_{x_R', x_R''}}$$
$$\times N[\hat{O}_{x_A''} \cdots \hat{O}_{x_Z''}]$$ ⟸ some contractions

from contractions

$$K_R' = w_A \cdots w_Z \, S_R \sum_{x_R', x_R''} d^{(R)}_{x_R' x_R''}$$

internal indices from contractions

external indices (uncontracted lines)

$$\times \, \hat{O}_{x_R''}$$

from uncontracted $X^\dagger, X$

$$K_A \cdots K_Z = \sum_R N_R K_R'$$

nonequivalent terms

\# of times one obtains K_R'

$$= \sum_R K_R \, , \text{ where}$$

$$K_R = N_R K_R'$$

$$K_R = s_R \underbrace{N_R w_A \cdots w_Z}_{} \times \sum_{\chi_R', \chi_R''} d^{(R)}_{\chi_R' \chi_R''} \hat{O}_{\chi_R''}$$

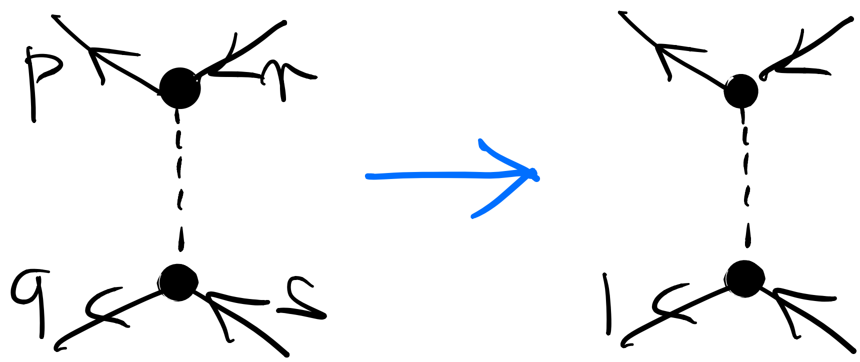
$$= s_R \underbrace{w_R}_{} \sum_{\chi_R', \chi_R''} d^{(R)}_{\chi_R' \chi_R''} \hat{O}_{\chi_R''}$$

This is exactly why we must have

$$w_R = N_R w_A \cdots w_Z .$$

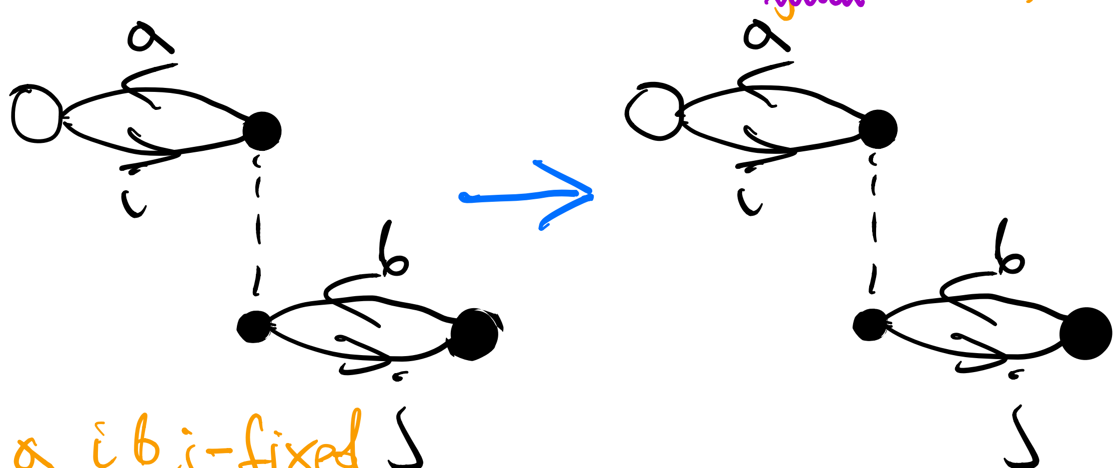
WEIGHTS

We need to talk about ORIENTED SKELETONS.



d-m (p,q,r,s-free)

skeleton (obtained by removing all free labels)



a, i, b, j-fixed d-m

skeleton

EQUIVALENCE CLASSES characterizing skeletons

We must identify equivalent elements and group them.

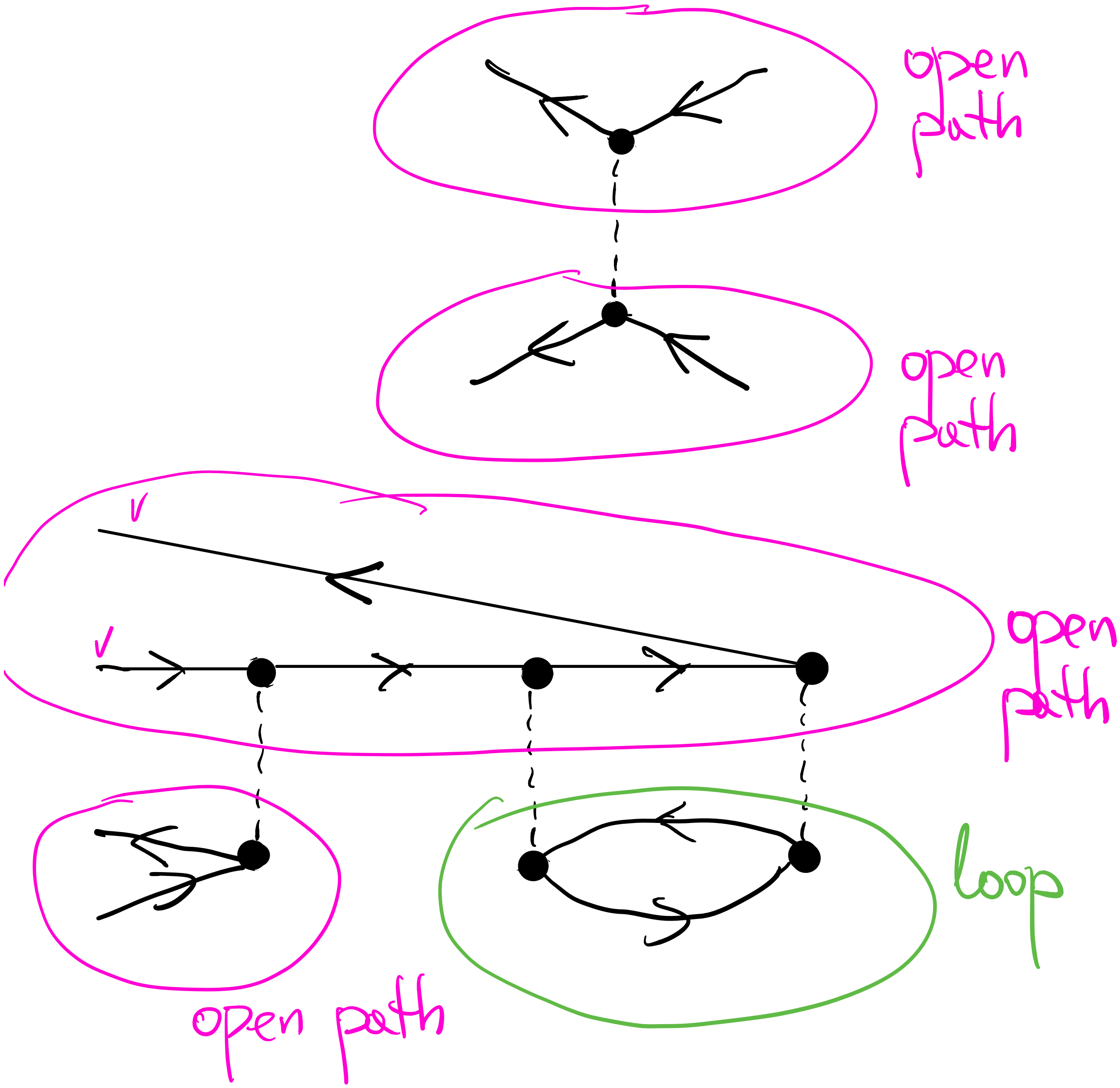
↑ depend on representation (Goldstone, Hugenholtz, ...)

Goldstone representation

elements ↳ ORIENTED PATHS ← sequencies of oriented adjacent lines ✓

↙ CLOSED LOOPS (no external lines)

↘ OPEN PATHS (have external lines)

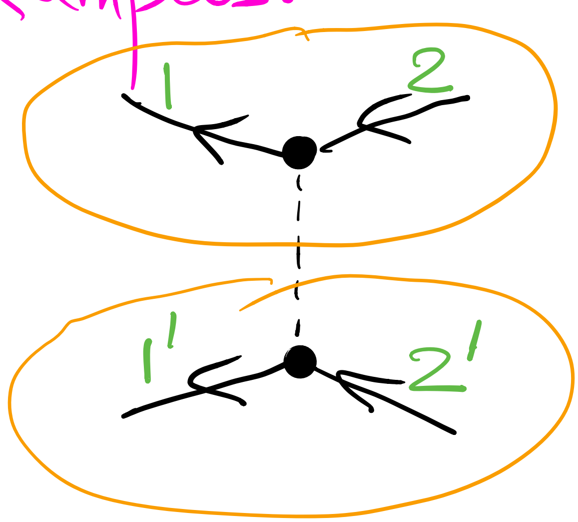


Two oriented paths are equivalent if there exists an automorphism which transforms these paths one onto another.

We will organize oriented paths in equivalence classes.

Examples:

1



$r = \#$ of equivalence classes

$n_\ell = \#$ of elements in class ℓ $(\ell = 1, \dots, r)$

$$\underbrace{\left[\underbrace{(1,2)}_{\text{open path}}, \underbrace{(1',2')}_{\text{open path}}\right]}_{\text{equivalence class}}$$

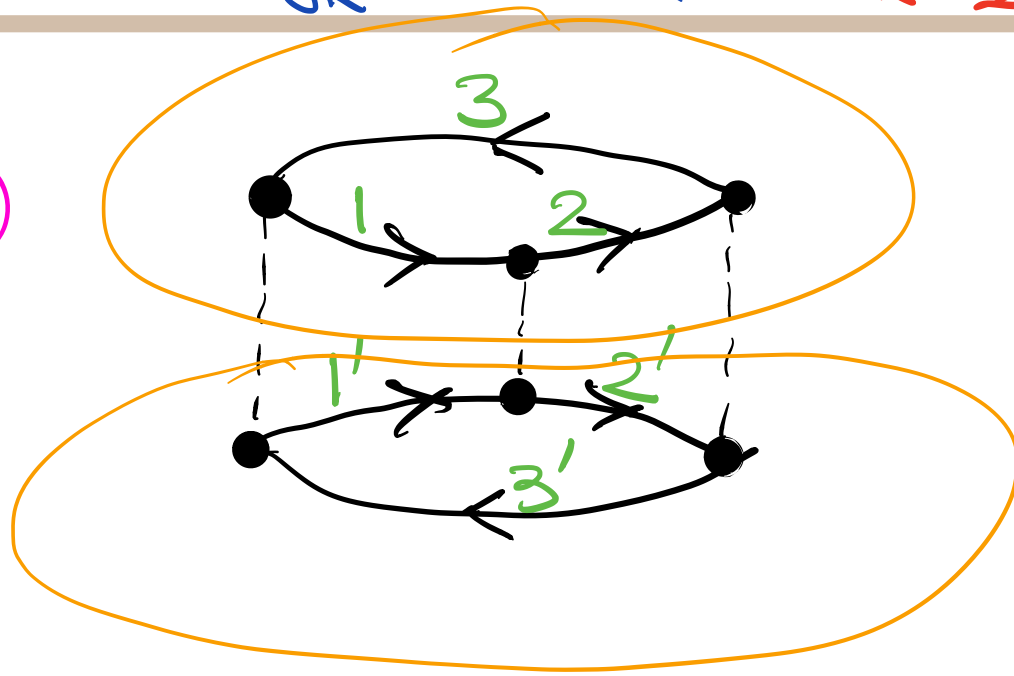
$r = 1, \; n_1 = 2$

These open paths are equivalent: $(11')(22')$

Γ_R: $e, (11')(22')$; $g_R = 2$

$g_R = 2! = 2$, $w_R = \frac{1}{2}$

2



$$\underbrace{[\underbrace{(1,2,3)}_{\text{loop}}, \underbrace{(1',2',3')}_{\text{loop}}]}_{\text{equivalence class}}$$

These two loops are equivalent: $(11')(22')(33')$

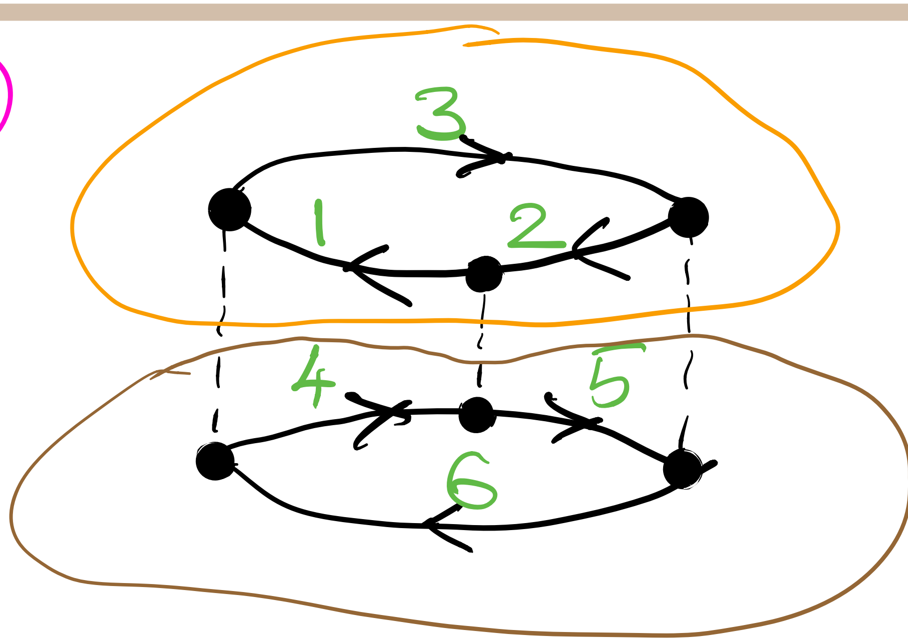
$r = 1$, $n_1 = 2$

Γ_R: $e, (11')(22')(33')$; $g_R = 2$

$g_R = 2! = 2$

$w_R = \frac{1}{2}$

3



$$\underbrace{[(1,2,3)]}_{\text{class}} ; \underbrace{[(4,5,6)]}_{\text{class}}$$

$$r=2, \; n_1=1, \; n_2=1$$

$$\Gamma_R : e \; ; \; g_R = 1 \qquad w_R = 1$$

$$g_R = 1! \cdot 1! = 1$$

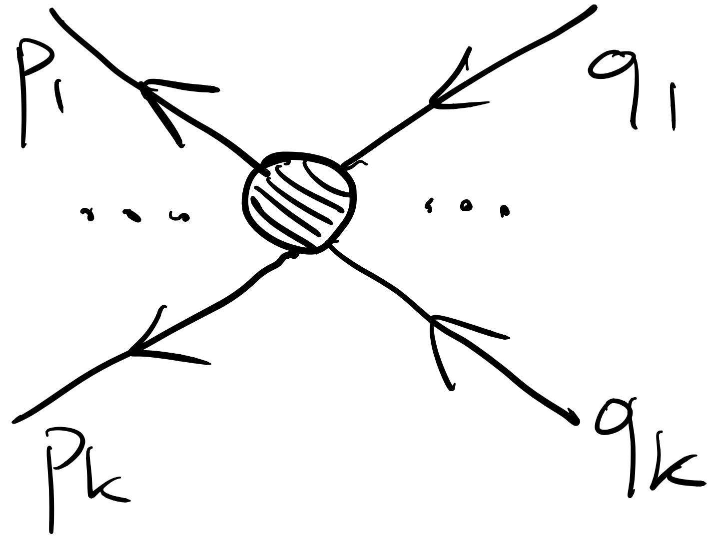
Hugenholtz d-ms

Used to represent operators using antisymmetrized matrix elements, e.g.,

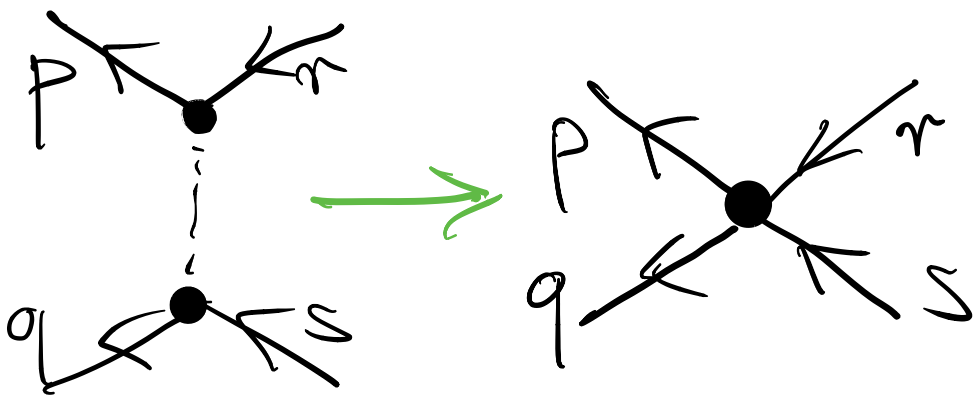
$$\hat{O}_k = \left(\frac{1}{k!}\right)^2 \times \sum_{\substack{p_1 \cdots p_k \\ q_1 \cdots q_k}} \langle p_1 \cdots p_k | \hat{o}_k | q_1 \cdots q_k \rangle_A$$

$$\times \; N[X^\dagger_{p_1} \cdots X^\dagger_{p_k} X_{q_k} \cdots X_{q_1}]$$

$\longleftarrow$ or $\longrightarrow$



Example:



Goldstone

$$\frac{1}{2}\sum_{pqrs} \langle pq|\hat{v}|rs\rangle \times N[X_p^\dagger X_r X_q^\dagger X_s]$$

Hugenholtz

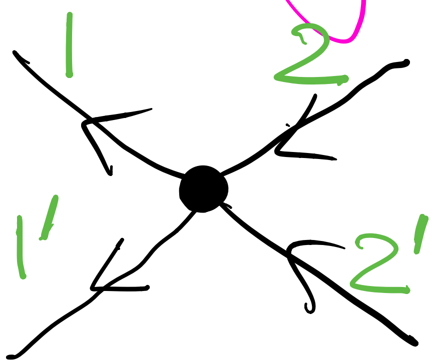
$$\frac{1}{4}\sum_{pqrs} \langle pq|\hat{v}|rs\rangle_A \times N[X_p^\dagger X_r X_q^\dagger X_s]$$

Hugenholtz representation

elements → FERMION LINES

Two Fermion lines are equivalent if they have the same orientation and originate from (or connect) the same vertices (and cannot carry fixed indices).

④ (analog of ①)



$\underbrace{[1,1']}_{\text{class}}$; $\underbrace{[2,2']}_{\text{class}}$

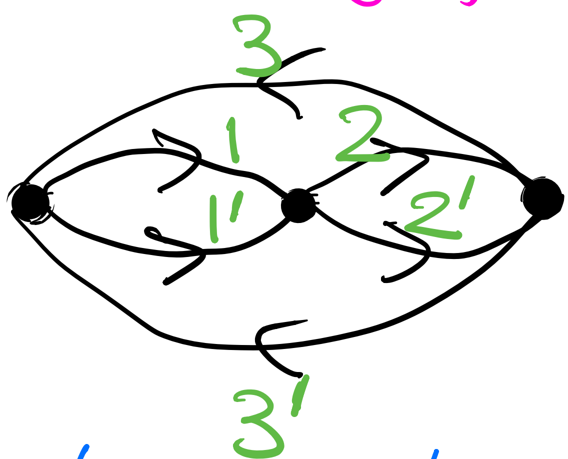
$r=2,\ n_1=2,\ n_2=2$

Γ_R: $e,\ (11'),\ (22'),\ (11')(22')$

$g_R=4,\ g_R=2!\cdot 2!=4$

$w_R=\frac{1}{4}$

5 (analog of 2)



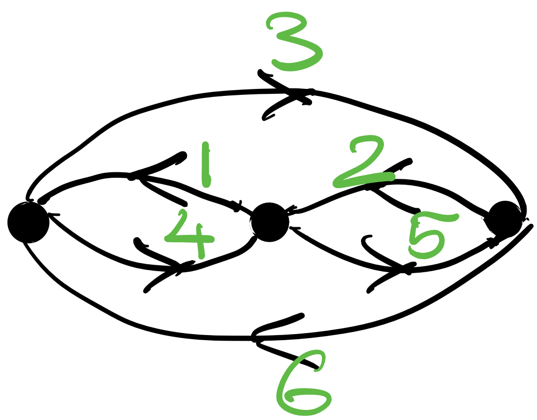
$[1,1'];\ [2,2'];\ [3,3']$
$\underbrace{}_{\text{class}}\ \underbrace{}_{\text{class}}\ \underbrace{}_{\text{class}}$

$r=3,\ n_1=n_2=n_3=2$

Γ_R: $e, (11'), (22'), (33'),$
$(11')(22'), (11')(33'), (22')(33'),$
$(11')(22')(33')$; $g_R=8$, $g_R=(2!)^3=8$

$w_R=\frac{1}{8}$

6 (analog of 3)



[1]; [2]; [3]; [4]; [5]; [6]

$r=6, \; n_i = 1, \; i=1,\dots,6$

$\Gamma_R : e \; ; \; g_R = 1,$

$g_R = (1!)^6 = 1$

$h_R = 1$

All automorphisms of skeleton R form a group of automorphisms, designated as Γ_R.

Weight of skeleton (or diagram) R is

$$W_R = g_R^{-1},$$

# of elements in Γ_R

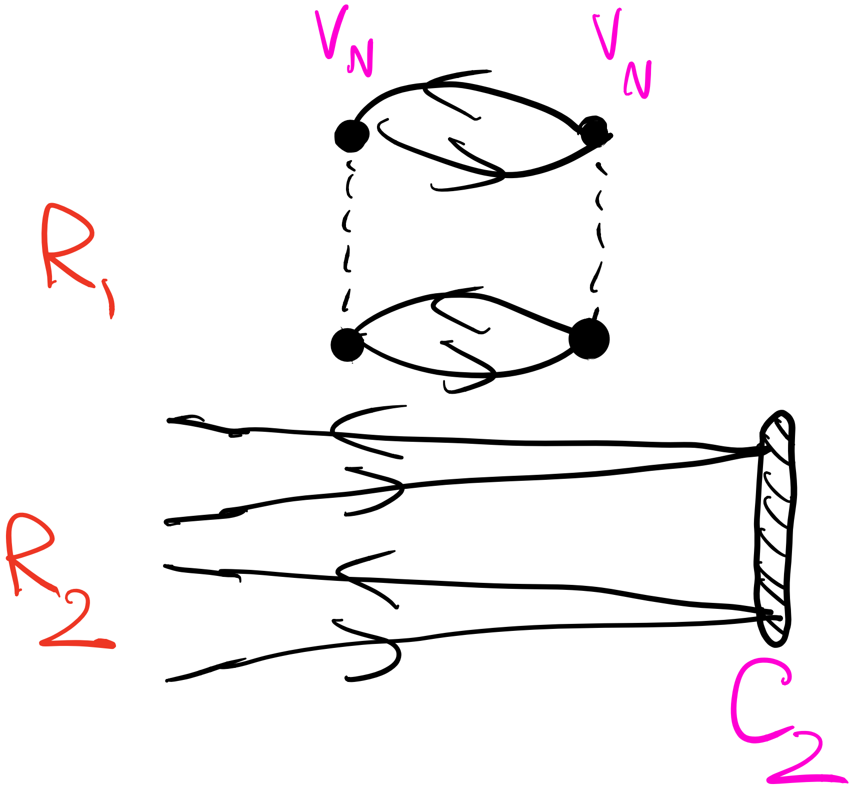
where g_R is the order of group Γ_R (topological factor).

$$g_R = \prod_{l=1}^{r} n_l!$$ ✓

So far we discussed
connected d-ms

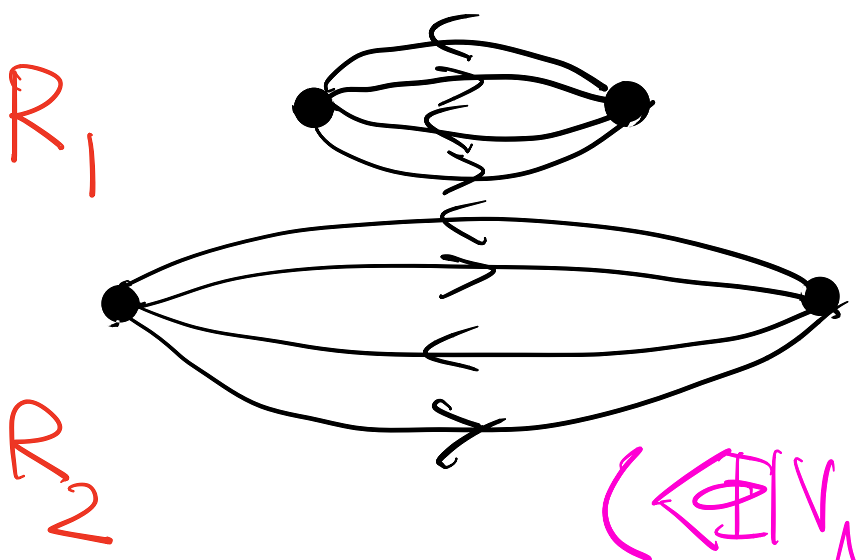
What about DISCONNECTED
D-MS ?

Examples:



$(V_N V_N C_2 |\Phi\rangle$

$R = R_1 + R_2$

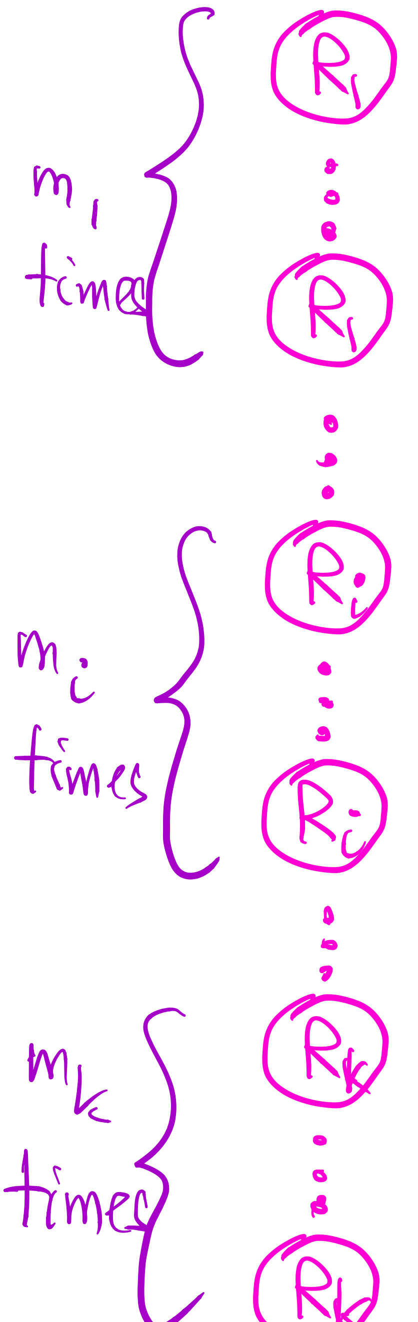


$R = R_1 + R_2$

$(\langle \Phi | V_N V_N V_N V_N | \Phi \rangle$

In general, $R = \sum_{i=1}^{k} m_i R_i$

$m_i \geq 1$

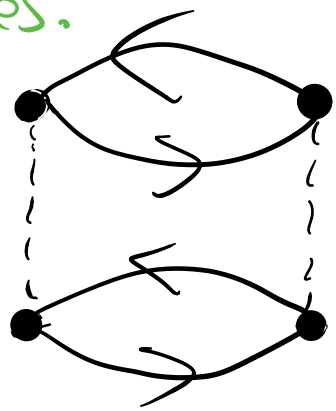


$$w_R = g_R^{-1}$$

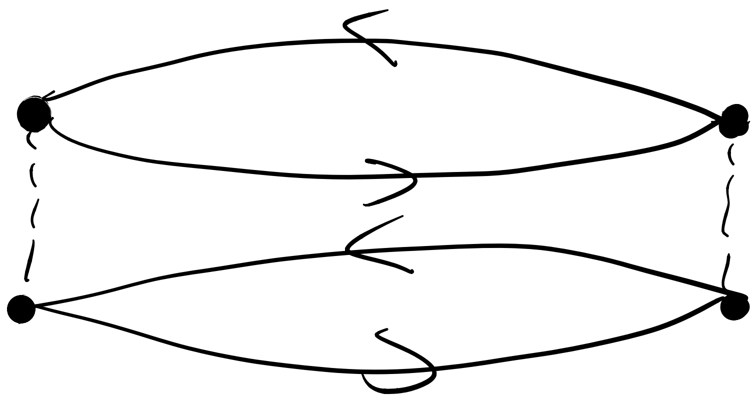
$$g_R = \prod_{i=1}^{k} m_i! \left(g_{R_i}\right)^{m_i}$$

Examples:

$R = R_1 + R_2$



R_1



R_2

$m_1 = 1, \; m_2 = 1, \; k = 2$

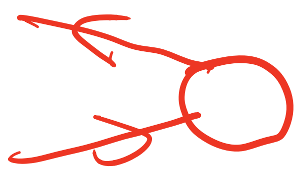
$g_{R_1} = 2, \; g_{R_2} = 2$

$$g_R = 1!(g_{R_1})^1 \, 1!(g_{R_2})^1$$
$$= 2 \cdot 2 = 4, \quad w_R = \frac{1}{4}$$

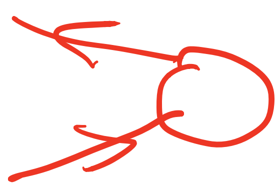
From CC:

$$e^{T}|\Phi\rangle, \quad T = T_1 + T_2 + \dots$$

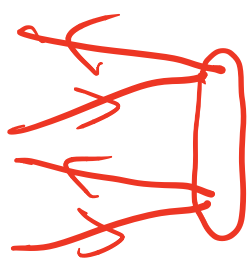
one of the terms → $\underbrace{\frac{1}{2!}}_{\sim} T_1^2 \, \frac{1}{2!} T_2^2 = \frac{1}{4} T_1^2 T_2^2$



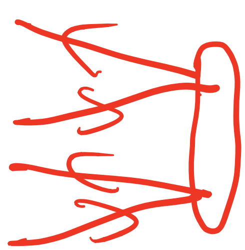
$\leftarrow T_1$ $\qquad g_{T_1} = 1$



$\leftarrow T_1$ $\qquad g_{T_2} = 2$



$\leftarrow T_2$ $\qquad m_1 = 2$, $\quad m_2 = 2$



$\leftarrow T_2$

$$g_R = m_1!(g_{T_1})^{m_1} m_2!(g_{T_2})^{m_2}$$
$$= 2! \cdot 1^2 \cdot 2! \cdot 2^2$$
$$= 16$$
$$w_R = \frac{1}{16}$$

Now, the LEMMA
(video 20)

Lemma:

Let A and B be the CONNECTED skeletons (or diagrams) with weights w_A and w_B, respectively. Let R be the resulting diagram obtained by connecting a certain number of oriented lines of A and B. Then, the number N_R of distinct ways of obtaining the skeleton (or diagram) R from A and B is given by the formula

$$N_R = \frac{w_R}{w_A w_B}, \quad \checkmark$$

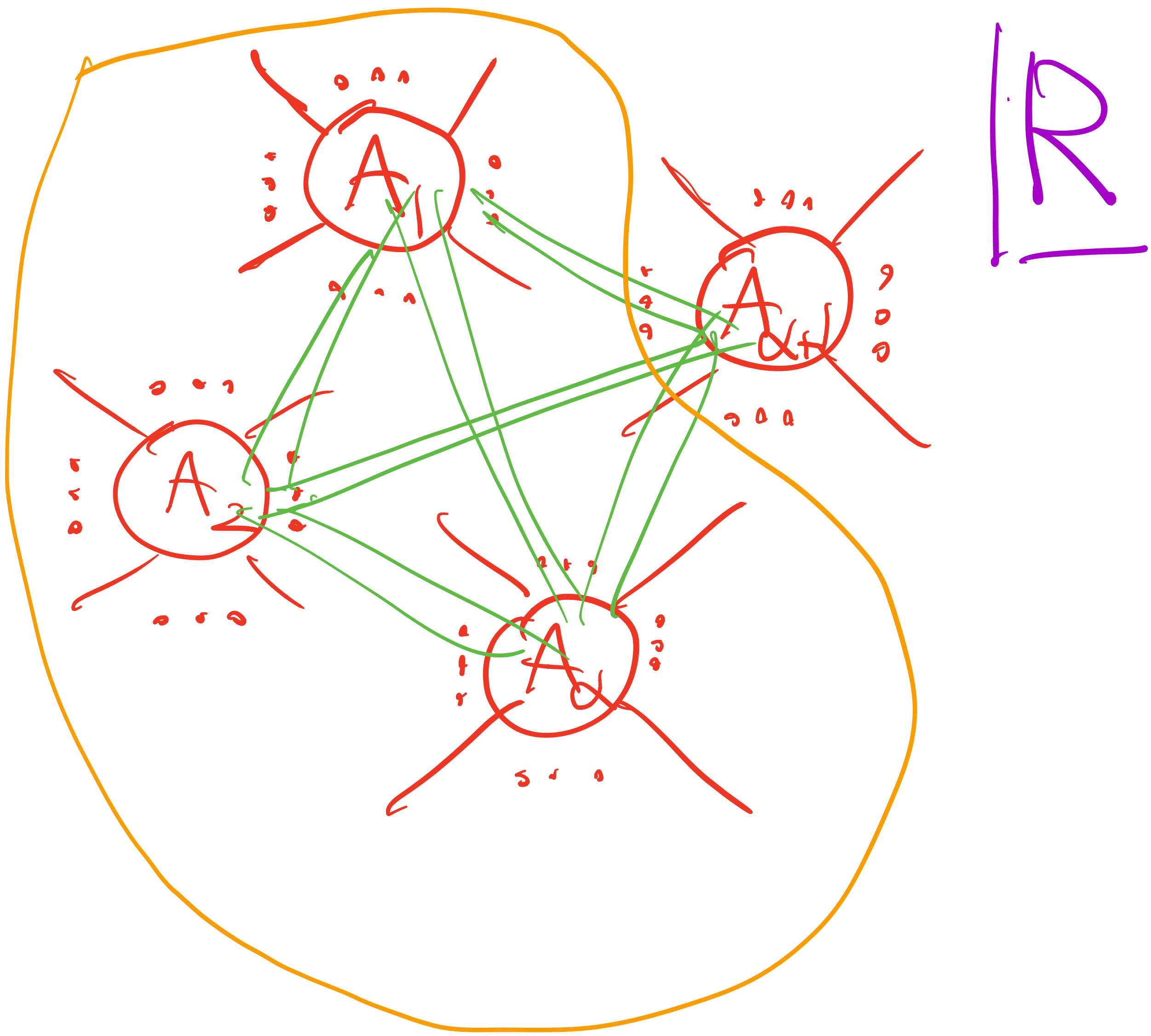
where w_R is the weight of R.

The proof is based on determining g_A, g_B, and g_R from equivalence classes of A, B, and R, and

independently calculating N_R.
One then shows that

$$N_R = \frac{g_A\, g_B}{g_R}.$$

Generalization (by induction)



Let us assume that the Lemma holds for α diagrams. We will extend it to $(\alpha+1)$ diagrams.

$$N_R = N_{A_1 \ldots A_\alpha} \; N_{(A_1 \ldots A_\alpha) A_{\alpha+1}}$$

# of distinct ways of forming $(A_1 \ldots A_\alpha)$

# of distinct ways of forming R from $(A_1 \ldots A_\alpha)$ and $A_{\alpha+1}$

$$N_{A_1 \ldots A_\alpha} \overset{\text{induction}}{\underset{\text{assumption}}{=}} \frac{W_{A_1 \ldots A_\alpha}}{W_{A_1} \ldots W_{A_\alpha}}$$

$$N_{(A_1 \ldots A_\alpha) A_{\alpha+1}} = \frac{W_R}{W_{A_1 \ldots A_\alpha} W_{A_{\alpha+1}}}$$

$$N_R = \frac{\cancel{W_{A_1 \ldots A_\alpha}}}{W_{A_1} \ldots W_{A_\alpha}} \cdot \frac{W_R}{\cancel{W_{A_1 \ldots A_\alpha}} W_{A_{\alpha+1}}}$$

$$= \frac{W_R}{W_{A_1} \ldots W_{A_\alpha} W_{A_{\alpha+1}}}$$