DIAGRAMMATIC METHODS

$K_A \, K_B \ldots K_Z \longrightarrow$ A B … Z

product of many-body operators in algebraic form

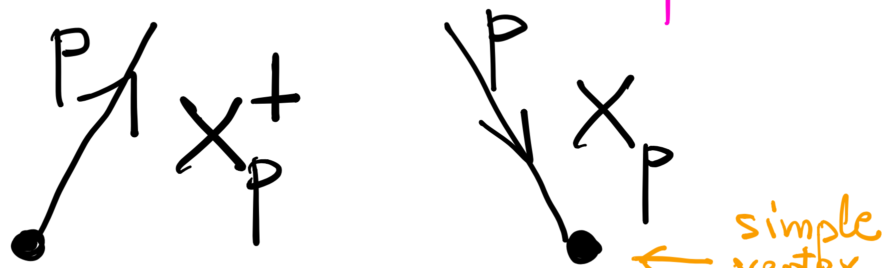
diagrams

$\longrightarrow \sum_R K_R \longrightarrow$ read diagrams to produce algebraic expressions

nonequivalent resulting diagrams

(admissible)

Fermion lines (oriented)



Combine simple vertices to form diagrams representing many-body operators.

This is done using nonoriented lines and additional symbols needed to distinguish between different many-body operators.

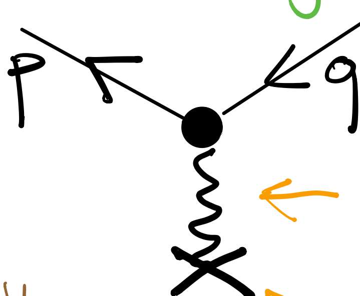
Our initial focus: GOLDSTONE DIAGRAMS

1-body operators (e.g., Z, Z_N, G, G_N, ...)

incoming incoming

$$Z = \sum_{p,q} \langle p|\hat{z}|q\rangle X_p^{\dagger} X_q$$

outgoing outgoing



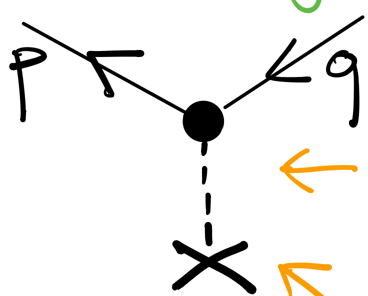
standard operator form

$\hat{z}$

one simple vertex with two fermion lines incoming incoming

$$Z_N = \sum_{p,q} \langle p|\hat{z}|q\rangle N[X_p^\dagger X_q]$$

outgoing outgoing



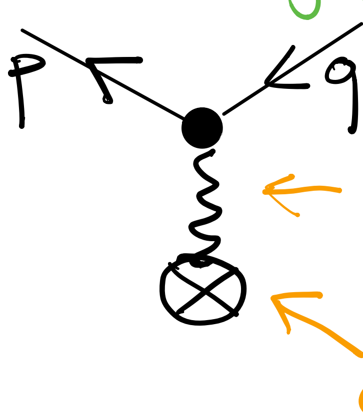
normal-ordered form

$\hat{z}$

incoming incoming

$$G = \sum_{p,q} \langle p|\hat{g}|q\rangle X_p^\dagger X_q$$

outgoing outgoing



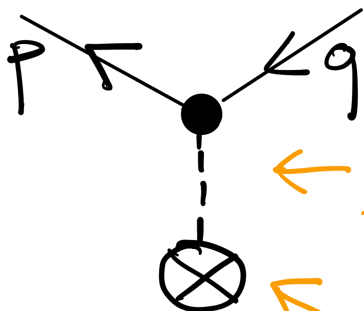
standard operator form

$\hat{g}$

incoming incoming

$$G_N = \sum_{p,q} \langle p|\hat{g}|q\rangle N[X_p^\dagger X_q]$$

outgoing outgoing



normal-ordered form

$\hat{g}$

In general,

scalar factor

$$\hat{O}_1 = \sum_{p,q} \langle p|\hat{o}_1|q\rangle X_p^\dagger X_q$$

or

$$N[X_p^\dagger X_q]$$

topological interpretation

$$\hat{O}_1 = w_{O_1} \sum_{\chi_{O_1}} d^{(O_1)}_{\chi_{O_1}} \hat{O}_{\chi_{O_1}}$$

w_{o_1} = weight = 1

x_{o_1} = indices of a diagram (p, q) [in this case, of external lines]

$d^{(o_1)}_{x_{o_1}}$ = scalar factor associated with a diagram $(\langle p|\hat{o}_1|q\rangle)$

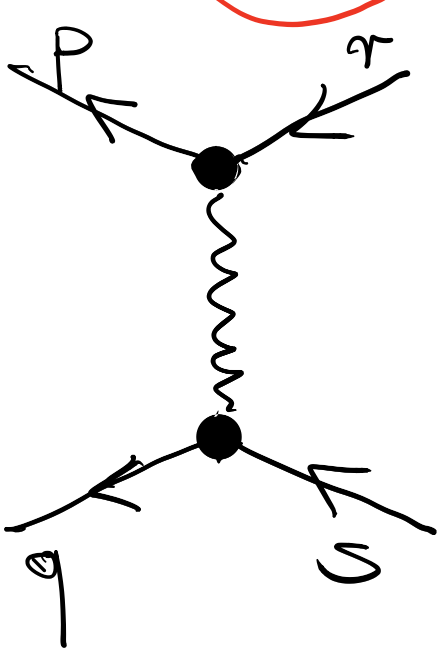
$\hat{O}_{x_{o_1}}$ = operator part of a diagram $(X_p^\dagger X_q$ or $N[X_p^\dagger X_q])$

2-body operators (V or $V_{N}, \ldots$)

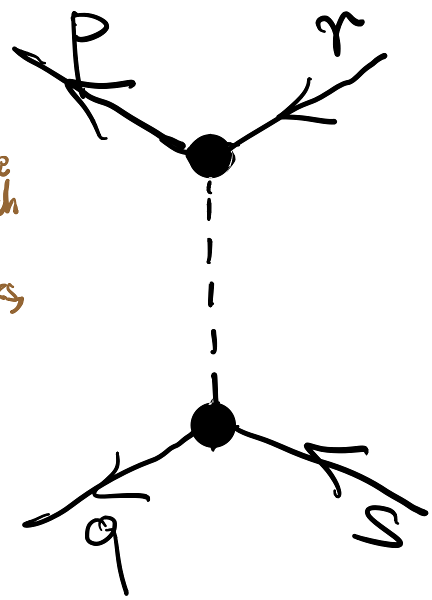
outgoing

$$V = \frac{1}{2}\sum_{p,q} \langle pq|\hat{v}|rs\rangle \times X_p^{\dagger} X_q^{\dagger} X_s X_r \qquad V_N = \frac{1}{2}\sum_{p,q} \langle pq|\hat{v}|rs\rangle \times N[X_p^{\dagger} X_r X_q^{\dagger} X_s]$$

incoming



two simple vertices, each having two fermion lines, bundled together



In general,

scalar factor

$$\hat{O}_2 = \frac{1}{2}\sum_{pqrs} \langle pq|\hat{o}_2|rs\rangle \times \begin{array}{c} X_p^{\dagger} X_q^{\dagger} X_s X_r \\ \text{or} \\ N[X_p^{\dagger} X_r X_q^{\dagger} X_s] \end{array}$$

Topologically,

$$\hat{O}_2 = w_{O_2} \sum_{\chi_{O_2}} d^{(O_2)}_{\chi_{O_2}} \hat{O}_{\chi_{O_2}}$$

scalar

operator part

weight

indices of a diagram (in this case, of external lines)

$$w_{O_2} = \frac{1}{2}, \quad d^{(O_2)}_{\chi} = \langle pq | \hat{O}_2 | rs \rangle$$

$$\chi_{O_2} = p, q, r, s$$

$$\hat{O}_{\chi_{O_2}} = X_p^+ X_q^+ X_s X_r \text{ or}$$

$$N[X_p^+ X_r X_q^+ X_s]$$

k-body operators:

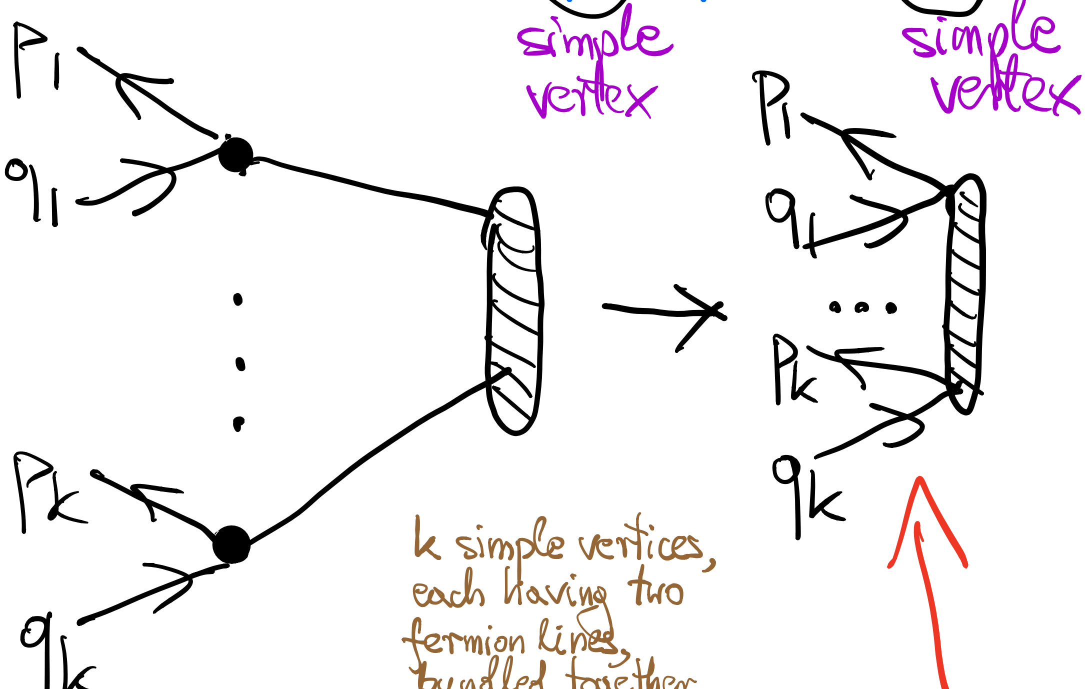
$$\hat{O}_k = \frac{1}{k!}\sum_{\substack{p_1,\dots,p_k\\ q_1,\dots,q_k}} \langle \overbrace{p_1\cdots p_k}^{\text{outgoing}}|\hat{o}_k|\overbrace{q_1\cdots q_k}^{\text{incoming}}\rangle$$

simple vertex

simple vertex

$$\times\; X^\dagger_{p_1}\cdots X^\dagger_{p_k} X_{q_k}\cdots X_{q_1}$$

$$\Big(\text{or } N\big[\underbrace{X^\dagger_{p_1}X_{q_1}}_{\text{simple vertex}}\cdots\underbrace{X^\dagger_{p_k}X_{q_k}}_{\text{simple vertex}}\big]\Big)$$



k simple vertices, each having two fermion lines, bundled together

$$\hat{O}_k = w_{O_k} \sum_{x_{O_k}} d^{(O_k)}_{x_{O_k}} \hat{O}_{x_{O_k}}$$

$$w_{O_k} = \frac{1}{k!} = \text{weight}$$

$$d^{(O_k)}_{x_{O_k}} = \langle \underbrace{p_1 \ldots p_k}_{\text{outgoing}} | \hat{O}_k | \overbrace{q_1 \ldots q_k}^{\text{incoming}} \rangle$$

$$x_{O_k} = p_1, \ldots, p_k, q_1, \ldots, q_k$$

$$\hat{O}_{x_{O_k}} = X^{\dagger}_{p_1} \ldots X^{\dagger}_{p_k} X_{q_k} \ldots X_{q_1}$$

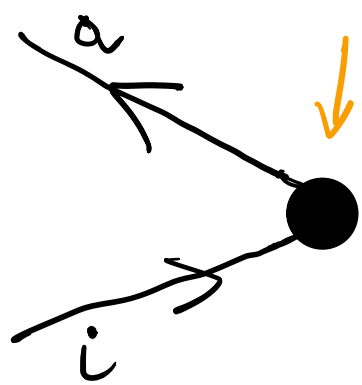
or

$$N[X^{\dagger}_{p_1} X_{q_1} \ldots X^{\dagger}_{p_k} X_{q_k}]$$

The above examples are involve FREE INDICES.
(summed over)

How about FIXED INDICES?

$$E^a_i = X_a^\dagger X_i \quad \left(= N[X_a^\dagger X_i]\right)$$



$d = 1$

$w_{E^a_i} = 1$ ← weight

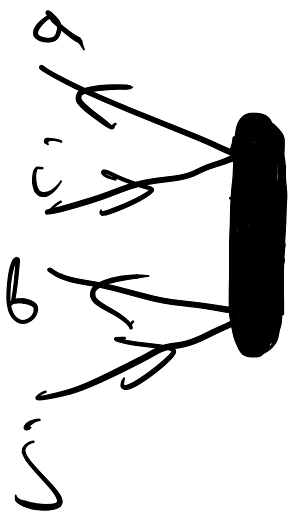
$\chi_{E^a_i} = a, i$ ← indices

$d^{(E^a_i)}_{\chi_{E^a_i}} = 1$ ← scalar factor

$\hat{O}_{\chi_{E^a_i}} = X_a^\dagger X_i$ (or $N[X_a^\dagger X_i]$)

↑ operator part

$$E^{ab}_{ij} = E^a_i E^b_j$$
$$= N[X^\dagger_a X_i X^\dagger_b X_j]$$



$w_{E^{ab}_{ij}} = 1$ ← weight

$d^{(E^{ab}_{ij})}_{a,b,i,j} = 1$ ← scalar factor

$\hat{O}_{ab,ij} = N[X^\dagger_a X_i X^\dagger_b X_j]$

↑ operator part

etc.

In general, all many body expressions (on the left & right-hand sides of WT) have the following form:

scalar factor

operator part

$$K_A = s_A \, w_A \sum_{x'_A, x''_A} d^{(A)}_{x'_A x''_A} \hat{O}_{x''_A}$$

sign

weight

indices of external fermion lines

indices of internal fermion lines

What do we do next?

$K_A \cdots K_Z$ ← operator product

$\downarrow \qquad \downarrow$

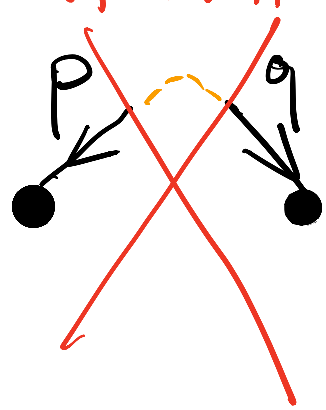
$A \cdots Z$ → manipulate d-ms by contractions to obtain resulting d-ms

diagrams on the lhs

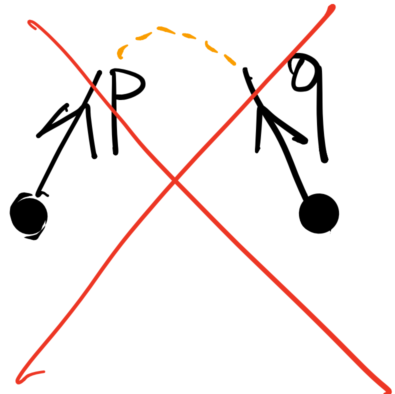
time line

CONTRACTIONS DONE DIAGRAMMATICALLY

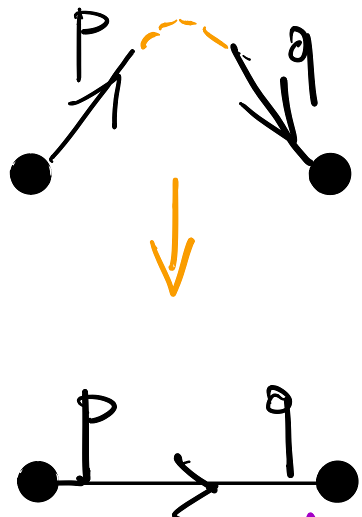
$\overbrace{X_p X_q} = 0$



$$\overline{X_p^{\dagger} X_q^{\dagger}} = 0$$

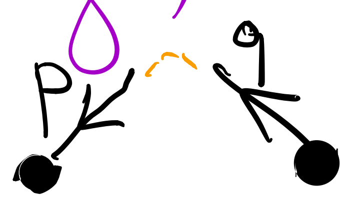


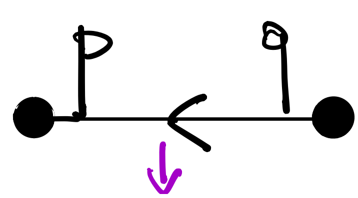
$$\overline{X_p^{\dagger} X_q} = \eta(p)\delta_{pq}$$



internal hole line (from left to right)

$$\overline{X_p X_q^{\dagger}} = \pi(p)\delta_{pq}$$



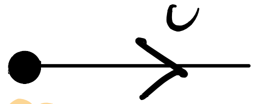


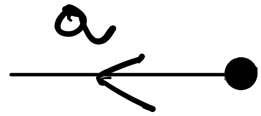
internal particle line
(from right to left)

X_p^+ $\quad$ X_p

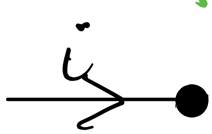
p-h character remains undefined
(lines drawn at an angle)

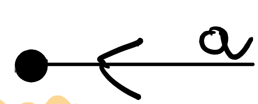
Lines drawn from left to right





$X_i^+ = Y_i$ $\quad$ $X_a^+ = Y_a^+$

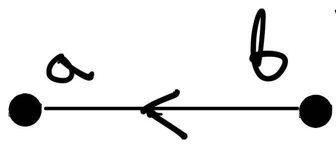




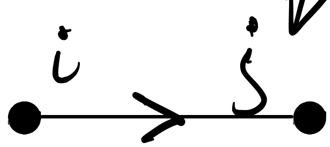
$X_i = Y_i^+$ $\quad$ $X_a = Y_a$

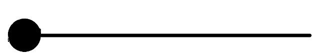
examples:

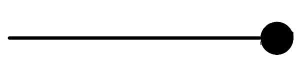
$$X_a X_b^\dagger = \delta_{ab}$$



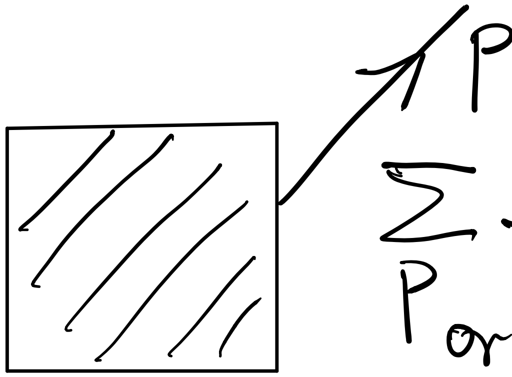
$$X_i^\dagger X_j = \delta_{ij}$$



 Y ✓

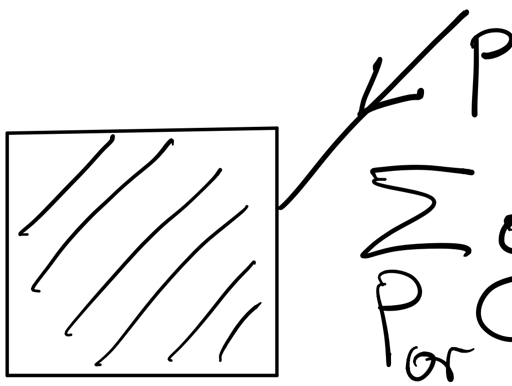
 $Y^\dagger$ ✓

Rules of contractions involving fixed and free indices (end of video 15, video 16)



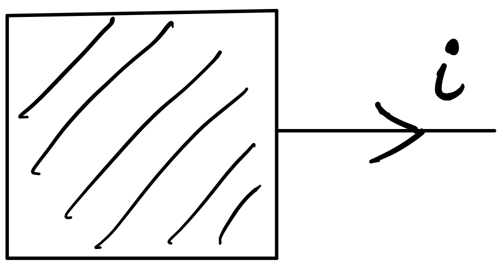
$$\sum_p f(\ldots p \ldots)\ldots x_p^{\dagger}\ldots$$

or fixed p



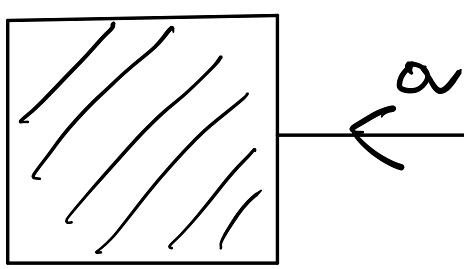
$$\sum_p g(\ldots p \ldots)\ldots x_p\ldots$$

or fixed p



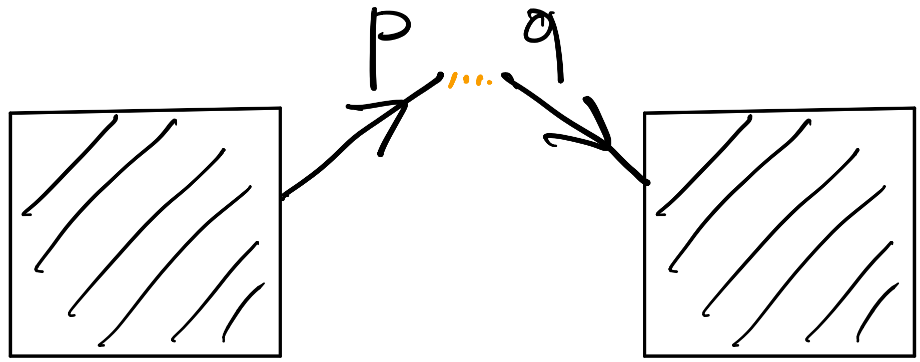
$\sum_i f(\ldots i \ldots) \ldots X^{\dagger}_{i\ldots}$

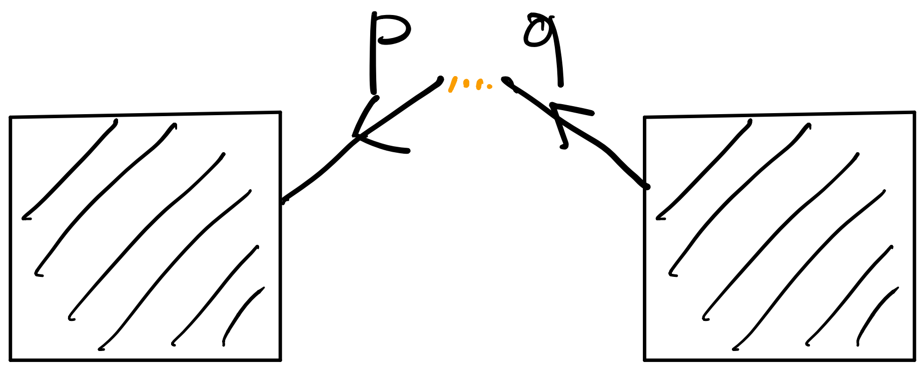
or fixed i



$\sum_a g(\ldots a \ldots) \ldots X_{a\ldots}$

or fixed a





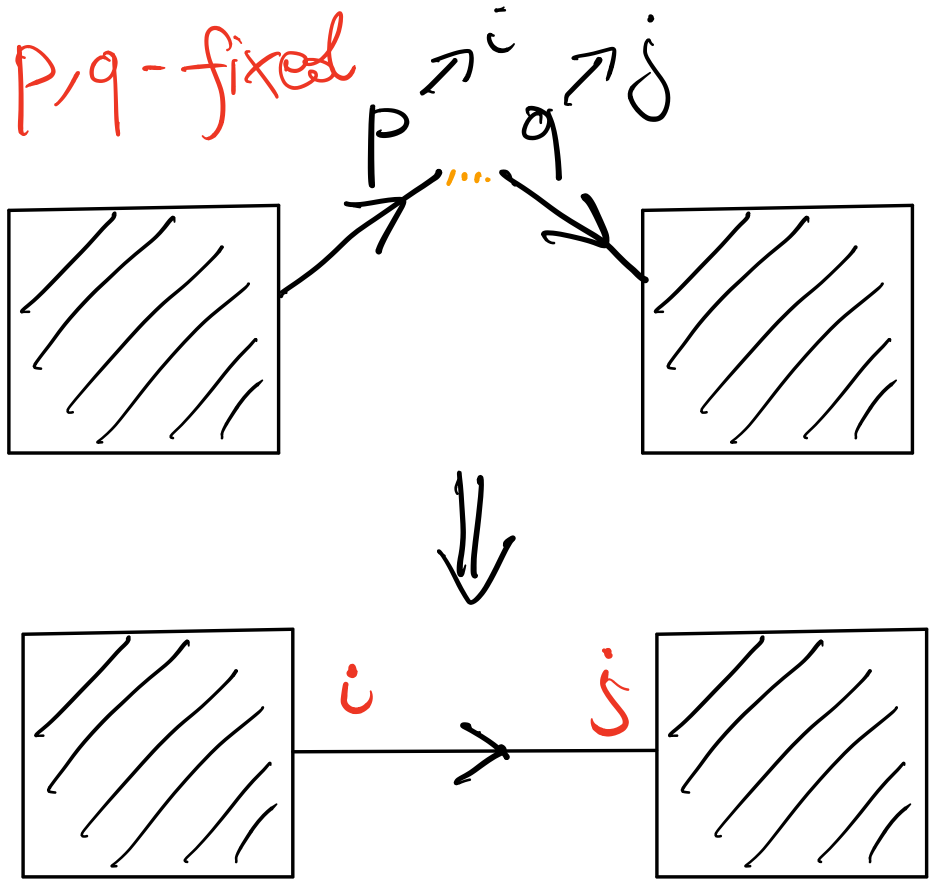
p,q – fixed

p-fixed, q – free

or

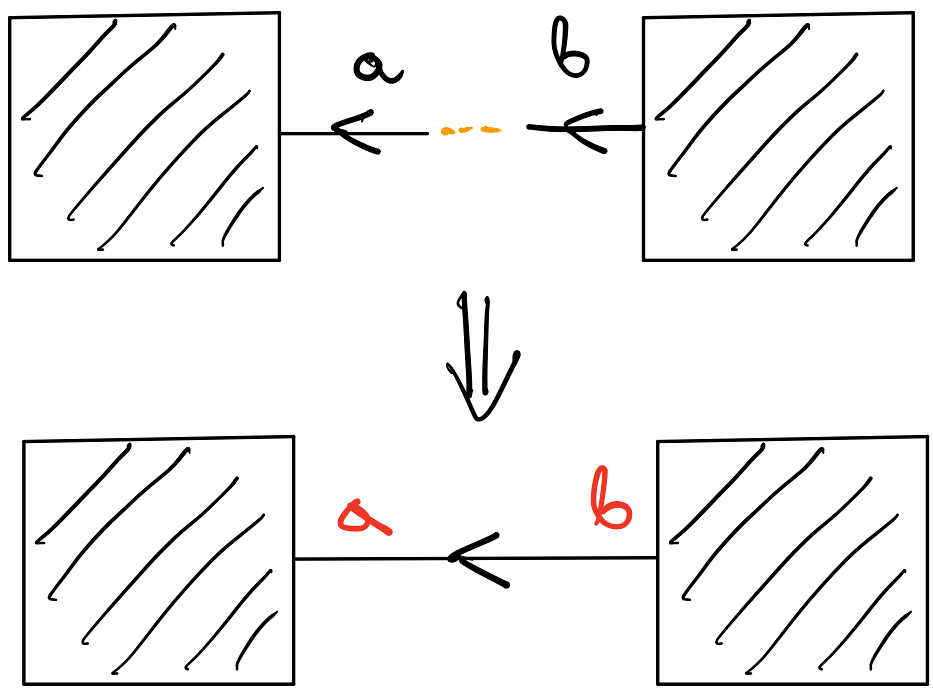
p-free, q-fixed

p,q – free



$$f(\dots i \dots)\, g(\dots j \dots) \times N[\dots \overbrace{X_i^{\dagger} \dots X_j} \dots]$$

$$= \pm \delta_{ij}\, f(\dots i \dots)\, g(\dots i \dots) \times N[\dots]$$

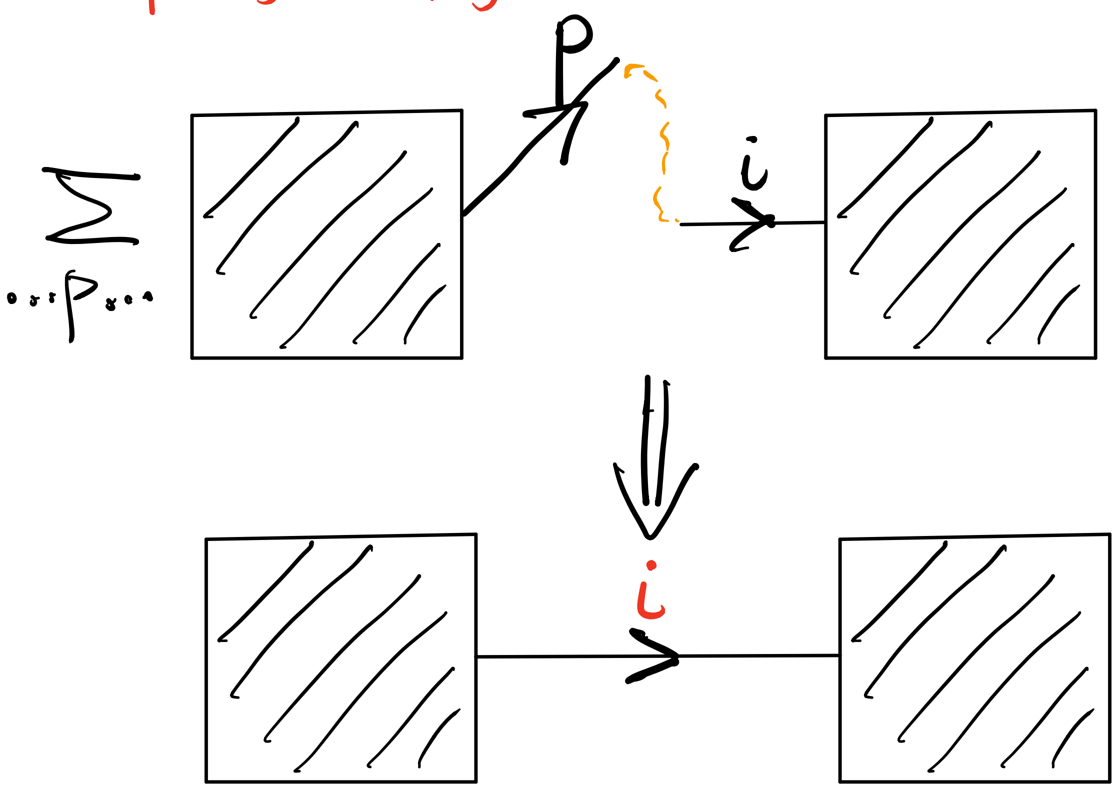


$$g(\ldots a \ldots)\, f(\ldots b \ldots) \times N[\ldots X_a \ldots X_b^{\dagger} \ldots] = \pm\, \delta_{ab}\, g(\ldots a \ldots)\, f(\ldots a \ldots)$$

Conclusion:

fixed + fixed = fixed, labeled by both indices, representing Kronecker delta.

p-free, q-fixed



$$\sum_{p} f(\ldots p \ldots)\, g(\ldots i \ldots) \times N[\ldots \overbracket{X_p^\dagger \ldots X_i} \ldots]$$

$$= \pm \sum_{\ldots p \ldots} \delta_{pi}\, f(\ldots i \ldots)\, g(\ldots i \ldots) \times N[\ldots]$$

$$= \pm\, f(\ldots i \ldots)\, g(\ldots i \ldots) \times N[\ldots]$$

Conclusion:

free + fixed = fixed, labeled by an index of a fixed line free + free = free (single summation index)

See video 16 for further details

$$K_A \ldots K_Z = \sum_R K_R \quad \checkmark$$

nonequivalent resulting d-ms

Admissible d-ms, example

$$\langle \Phi | K_A \ldots K_Z | \Phi \rangle$$

$$= \sum_R \langle \Phi | K_R | \Phi \rangle$$

fully contracted

d-ms with no external lines

$$K_A \cdots K_Z |\Phi\rangle = \sum_R \underbrace{K_R |\Phi\rangle}_{\text{annihilators } Y \text{ are fully contracted}}$$

draw only those d-ms that have external lines extending to the left

SEE VIDEO 16 FOR FURTHER DETAILS

Examples to develop intuitions :

① $\langle \Phi | Z_N | \Phi_i^a \rangle$

② $\langle \Phi_i^a | Z_N | \Phi_j^b \rangle$

③ $\langle \Phi_i^a | V_N | \Phi_j^b \rangle$ ✓

④ $\langle \Phi | V_N C_2 | \Phi \rangle$

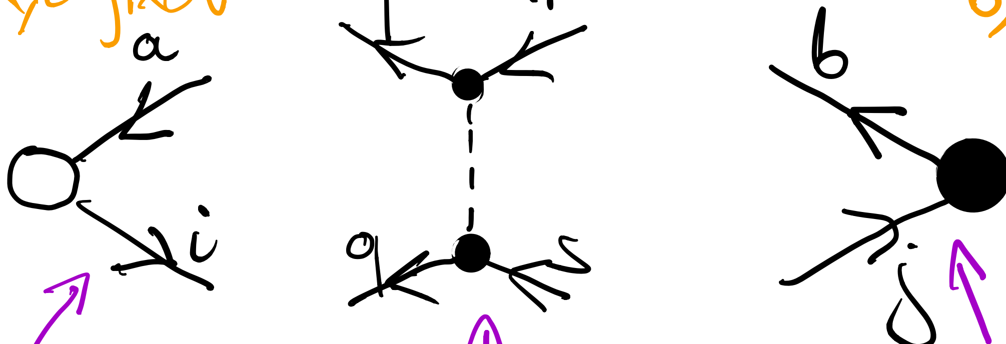
⑤ $Z_N | \Phi_i^a \rangle$

③ $\langle \Phi_i^a | V_N | \Phi_j^b \rangle$

$= \langle aj | \hat{v} | ib \rangle_A$

$\langle \Phi | (E_i^a)^\dagger \, V_N \, E_j^b | \Phi \rangle$ ✓

a,i-fixed b,j-fixed



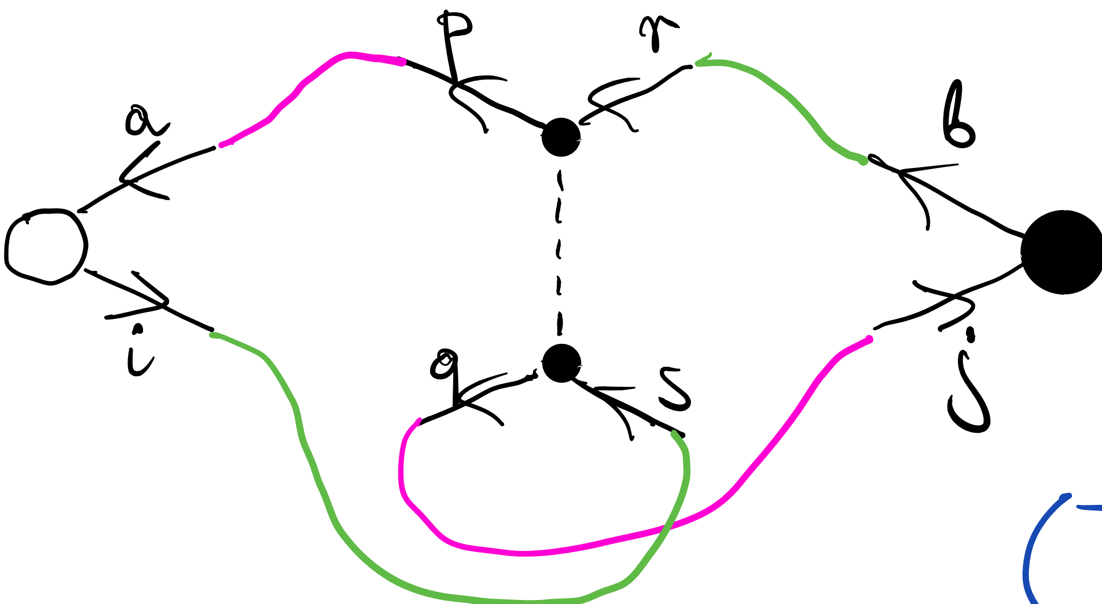
$(E_i^a)^\dagger = N[X_i^\dagger X_a]$ $E_j^b = N[X_b^\dagger X_j]$

$$V_N = \frac{1}{2} \sum_{pqrs} \langle pq | \hat{v} | rs \rangle \times N[X_p^\dagger X_r X_q^\dagger X_s]$$

p,q,r,s-free

We must draw resulting d-ms
WITH NO EXTERNAL LINES.

a, i - fixed $\quad p, q, r, s$ - free $\quad b, j$ - fixed

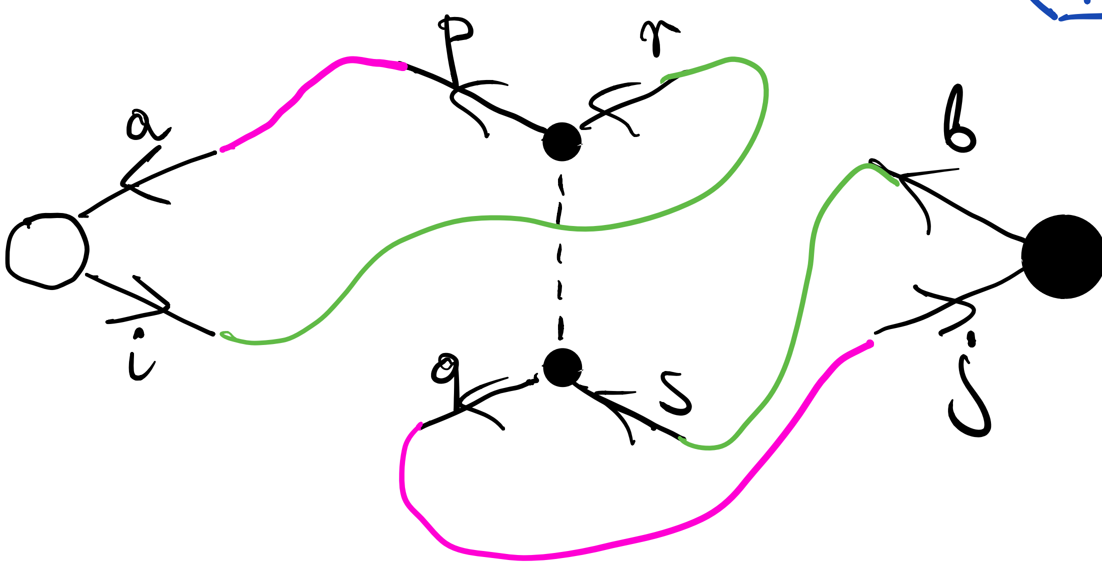


(I)

$$\frac{1}{2}\sum_{p,q,r,s} \langle pq|\hat{o}|rs\rangle \times N[X_i^{\dagger} X_a X_p^{\dagger} X_r X_q^{\dagger} X_s X_b^{\dagger} X_j]$$

$$= -\frac{1}{2}\langle aj|\hat{o}|bi\rangle$$

(II)

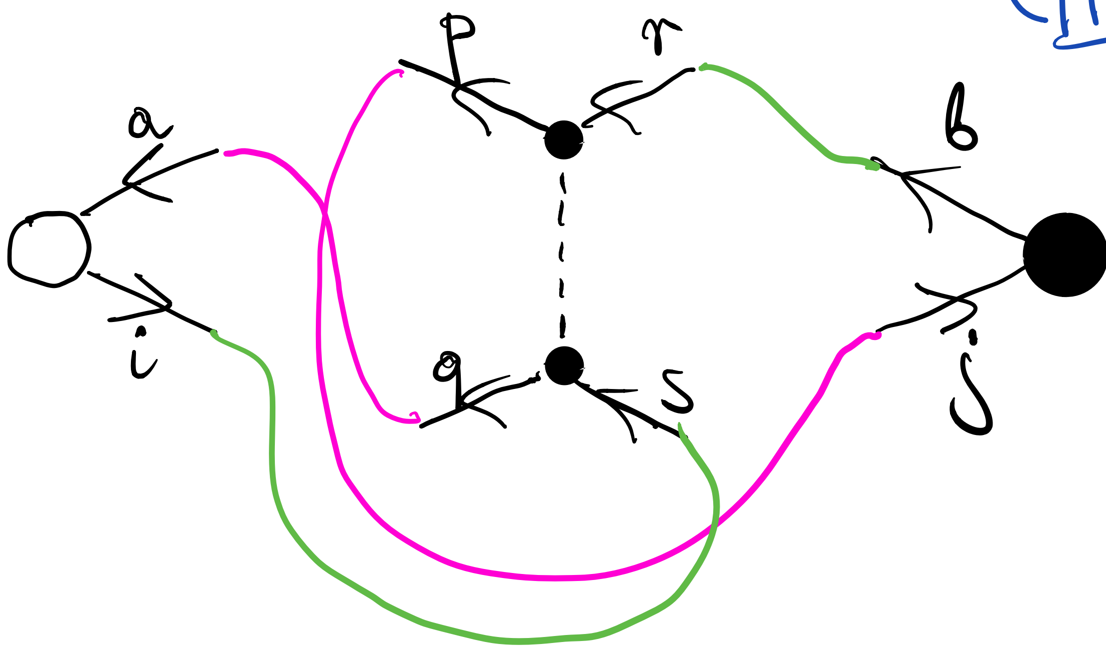


$$\frac{1}{2}\sum_{p,q,r,s} \langle pq|\hat{v}|rs\rangle$$

$$\times N[X_i^\dagger X_a X_p^\dagger X_r X_q^\dagger X_s X_b^\dagger X_j]$$

$$= +\frac{1}{2}\langle aj|\hat{v}|ib\rangle$$

(III)



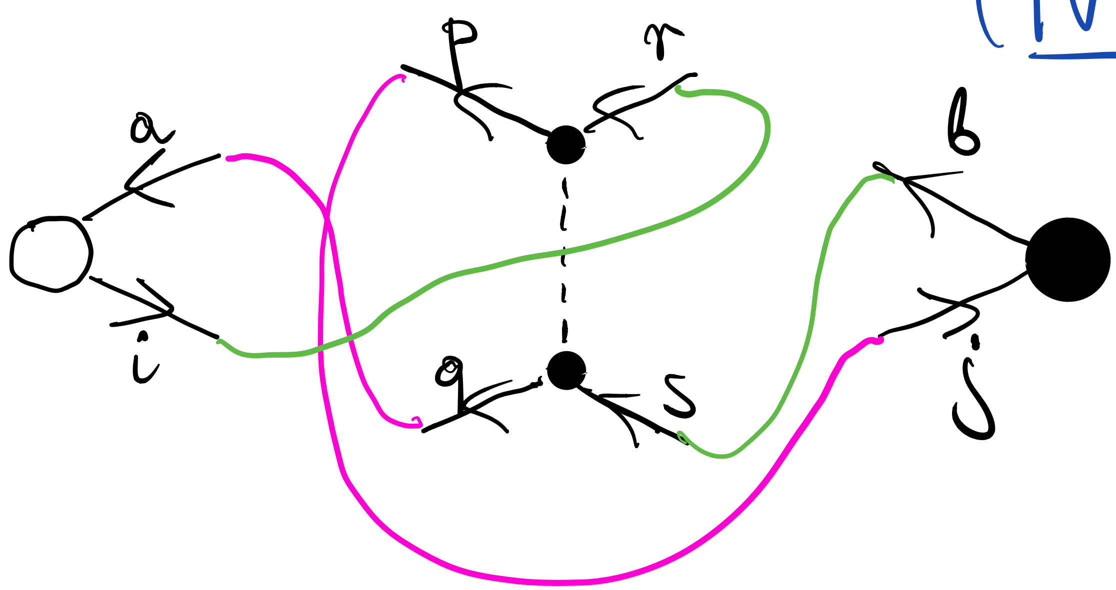
$$\frac{1}{2}\sum_{p,q,r,s} \langle pq|\hat{v}|rs\rangle$$

$$\times N[X_i^\dagger X_a X_p^\dagger X_r X_q^\dagger X_s X_b^\dagger X_j]$$

$$= +\frac{1}{2}\langle ja|\hat{v}|bi\rangle$$

$$\left(= +\frac{1}{2}\langle aj|\hat{v}|ib\rangle\right)$$

(IV)



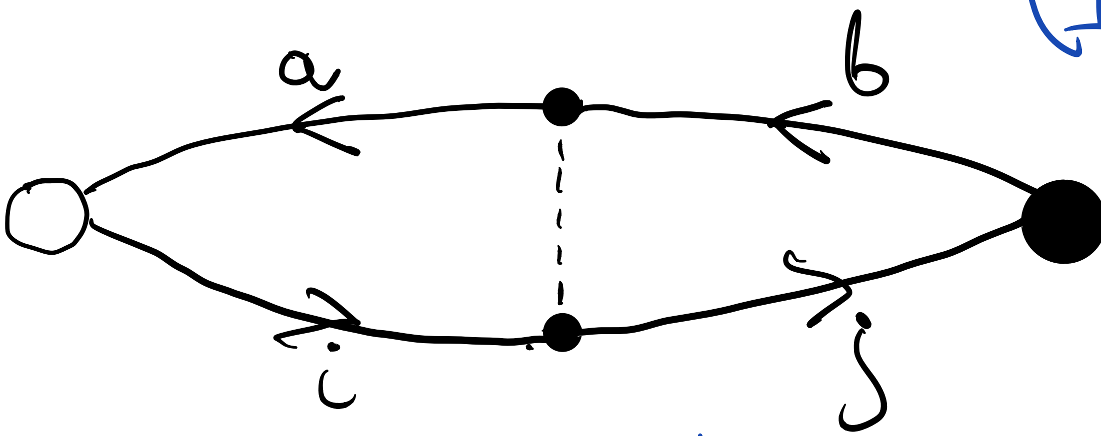
$$\frac{1}{2}\sum_{pq,r,s} \langle pq|\hat{v}|rs\rangle$$

$$\times N[X_i^\dagger X_a X_p^\dagger X_r X_q^\dagger X_s X_b^\dagger X_j]$$

$$= -\frac{1}{2}\langle ja|\hat{v}|ib\rangle$$

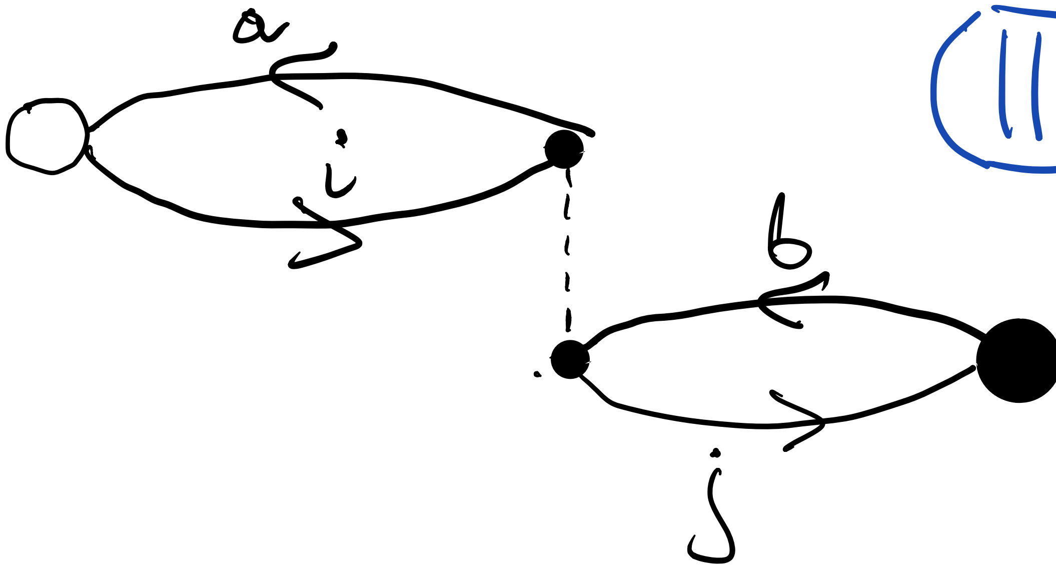
$$\left(= -\frac{1}{2}\langle aj|\hat{v}|bi\rangle\right)$$

(I)



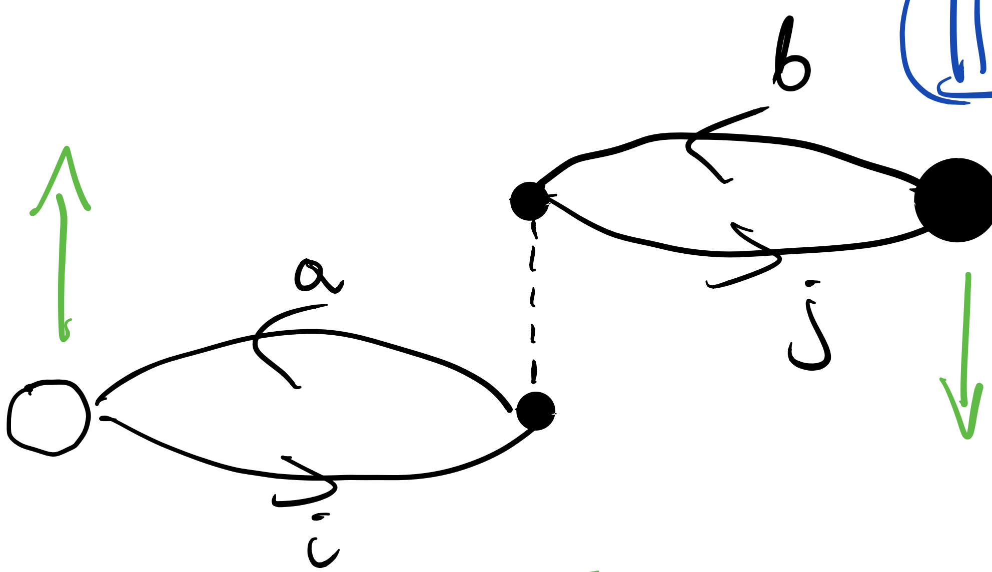
$$-\frac{1}{2}\langle aj|\hat{v}|bi\rangle$$

(II)



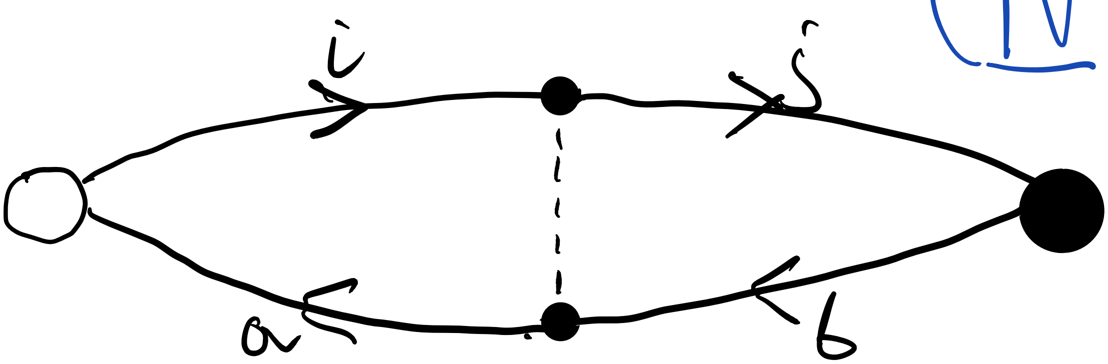
$$+\frac{1}{2}\langle aj|\hat{v}|ib\rangle$$

(III)



$$+\frac{1}{2}\langle ja|\hat{v}|bi\rangle = \frac{1}{2}\langle aj|\hat{v}|ib\rangle$$

(IV)



$$-\frac{1}{2}\langle ja|\hat{v}|ib\rangle = -\frac{1}{2}\langle aj|\hat{v}|bi\rangle$$

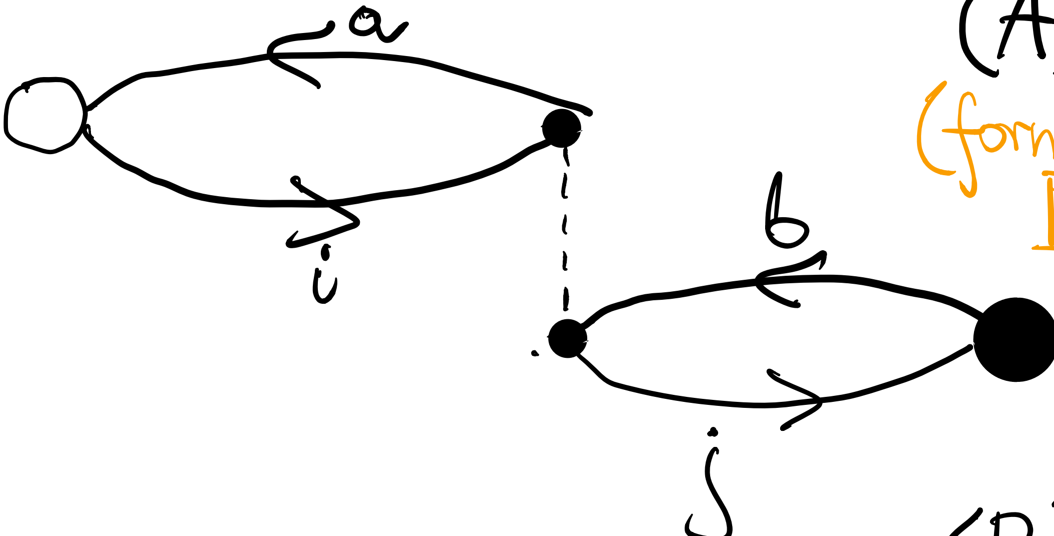
(II) = (III) ← equivalent

(I) = (IV) ← equivalent

Claim: Only non-equivalent resulting d-ms are needed

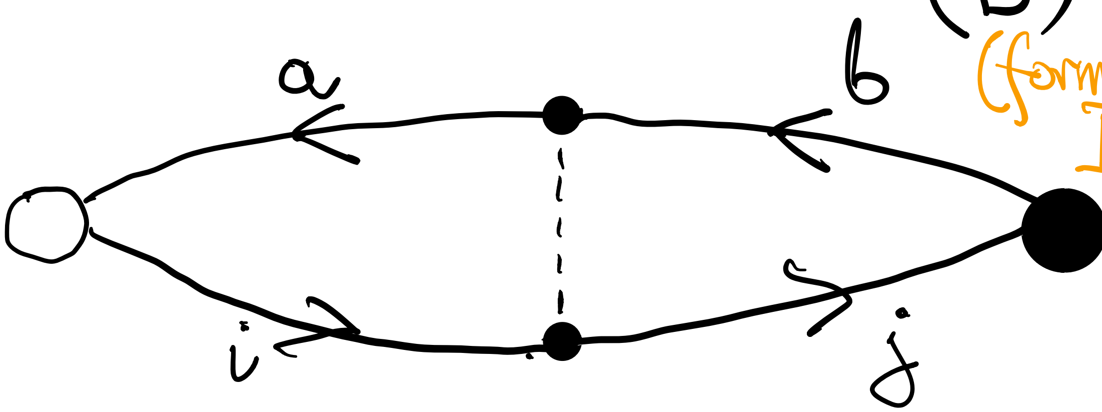
(A)

(formerly II)



(B)

(formerly I)

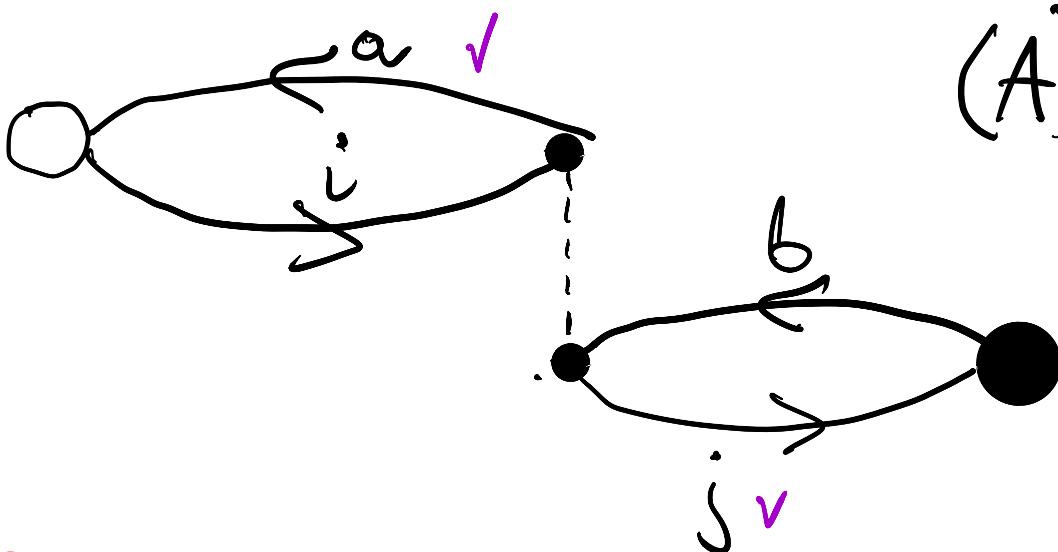


We want to obtain $\langle aj|\hat{v}|ib\rangle_A$ from (A) & (B).

This will happen if

$(A) = \langle aj|\hat{v}|ib\rangle$ ✓✓ (no $\frac{1}{2}$!)

$(B) = -\langle aj|\hat{v}|bi\rangle$ ✓✓ (no $\frac{1}{2}$!)



(A)

✓ $K_R = S_R h_R \sum_{\chi_R'} d^{(R)}_{\chi_R'}$, R=A,B

← indices of internal lines

$R = A$:

for $(E_i^a)^\dagger$

scalar factor of the resulting d-m is a product of scalar factors of component vertices

from E_j^b

outgoing

$$d_R = 1 \cdot \langle \underbrace{a_j}_{\text{outgoing}} | \hat{v} | \underbrace{i_b}_{\text{incoming}} \rangle \cdot 1 = \langle a_j | \hat{v} | i_b \rangle$$

incoming

from V vertex

$$w_R = N_R \, w_{(E_i^a)^\dagger} \, w_{V_N} \, w_{E_j^b}$$

$$= 2 \cdot 1 \cdot \frac{1}{2} \cdot 1 = 1$$

$$s_R = (-1)^{l+h}$$

$l = 2,\ h = 2$

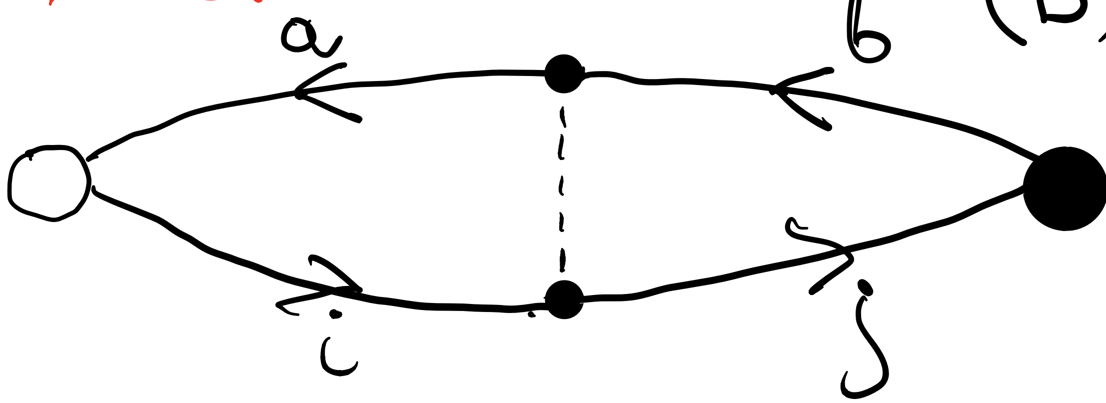
$$s_R = (-1)^{2+2} = +1$$

l = # of loops

h = # of internal hole lines

R=B:

(B)



$$d_R = 1 \cdot \langle aj|\hat{v}|bi\rangle \cdot 1$$

$$= \langle aj|\hat{v}|bi\rangle$$

$$w_R = \underset{N_R}{2} \cdot \underset{w_{(E_i^a)^\dagger}}{1} \cdot \underset{w_{V_N}}{\frac{1}{2}} \cdot \underset{w_{E_j^b}}{1} = 1$$

$$s_R = (-1)^{l+h} = -1$$

$$l=1, h=2$$

$$(A) + (B) = \langle aj | \hat{o}^{j} | ib \rangle_A$$

$$K_A \dots K_Z = \sum_R K_R$$

nonequivalent resulting d-ms

In order for this to be true, the following must hold:

$$N_R \, w_A \dots w_Z = w_R$$

$$\left(N_R = \frac{w_R}{w_A \dots w_Z} \right)$$

For the remaining examples, watch the videos (the end of video 16 and videos 17 and 18)