**Department of Chemistry, Michigan State University**

**CEM 991, Fall 2023**

**Midterm Exam I**

**Deadline for Submitting Solutions to the Instructor:** November 13, 2023, 4:10 pm EDT

**Name** (Last, First, Middle Initial): SOLUTIONS (Piecuch, Piotr)
**Student ID Number**:

## INSTRUCTIONS

- The maximum number of points that you can earn in this exam is 25. 25 points earned in this exam are equivalent to 250 points out of 1000 points constituting the final grade (i.e., 25 % of the final grade). In other words, 1 point earned in this exam is equivalent to 1 % of the final grade.
- Check that your booklet contains all 18 pages.
- Write your name and student ID number in the spaces provided above.
- Answer in the spaces provided. Use the extra pages 16-18 for additional space, if necessary. If you have technical difficulties with directly using this exam booklet, write down your solutions, as you normally do in the case of homework assignments, but do not forget to write your name and student ID number on the first page of your exam. Return your exam (this booklet or your own pages) to the instructor using the deadline described above. Late exams will not be accepted.
- Show ALL your work. Insufficient justification will result in a loss of points.
- Do not write anything in the score table given below.

| Problem | Score |
|---|---|
| 1 | /6 |
| 2 | /6 |
| 3 | /4 |
| 4 | /5 |
| 5 | /4 |
| Total | /25 |

1. (6 pts.) Consider a classical particle with mass $m$ under the influence of a force derived from the potential $V(\mathbf{r}, t)$ (no constraints). Let $x_1$, $x_2$, and $x_3$ define the Cartesian components of the radius vector $\mathbf{r}$ from the origin to the particle and let $\rho$, $\phi$, and $z$ designate the cylindrical components of $\mathbf{r}$, so that

$$\begin{aligned} x_1 &= \rho\cos\phi, \\ x_2 &= \rho\sin\phi, \\ x_3 &= z. \end{aligned}$$

(a) Derive the formula for the kinetic energy of the particle in cylindrical coordinates (3 pts.).

(b) Derive the explicit formula for the Hamilton function (Hamiltonian) of the particle in cylindrical coordinates assuming that $V = \frac{1}{2}K\rho^2$ ($K$ is a constant) (3 pts.).

*Solution*

(a)

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

$$\dot{x}_1 = \frac{dx_1}{dt} = \dot{\rho}\cos\varphi + \rho(-\sin\varphi)\dot{\varphi}$$
$$= \dot{\rho}\cos\varphi - \rho\sin\varphi\,\dot{\varphi}$$

$$\dot{x}_2 = \frac{dx_2}{dt} = \dot{\rho}\sin\varphi + \rho\cos\varphi\,\dot{\varphi}$$

$$\dot{x}_3 = \dot{z}$$

$$T = \frac{m}{2}\left[(\dot{\rho}\cos\varphi - \rho\sin\varphi\,\dot{\varphi})^2 + (\dot{\rho}\sin\varphi + \rho\cos\varphi\,\dot{\varphi})^2 + \dot{z}^2\right]$$

$$= \frac{m}{2}\Big[\dot{\rho}^2\cos^2\varphi \cancel{- 2\rho\dot{\rho}\sin\varphi\cos\varphi\,\dot{\varphi}} + \rho^2\sin^2\varphi\,\dot{\varphi}^2$$
$$+ \dot{\rho}^2\sin^2\varphi \cancel{+ 2\rho\dot{\rho}\sin\varphi\cos\varphi\,\dot{\varphi}} + \rho^2\cos^2\varphi\,\dot{\varphi}^2$$
$$+ \dot{z}^2\Big] = \frac{m}{2}\left[\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right]$$

Thus, $\boxed{T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right)}$.

(b) Since there are no constraints and force is derived from a potential,

$$H(\underline{q}, \underline{p}, t) = T(\underline{q}, \underline{p}) + V(\underline{q}, t).$$

In our case,

$$\underline{q} = (\rho, \varphi, z), \quad \underline{p} = (p_\rho, p_\varphi, p_z).$$

In general, $p_\ell = \frac{\partial L}{\partial \dot{q}_\ell}$, where $L = T - V$.

In our case, since $V$ does not depend on $\dot{q}_\ell$ $(\dot{\rho}, \dot{\varphi}, \dot{z})$, we have

$$p_\ell = \frac{\partial T}{\partial \dot{q}_\ell} - \frac{\partial V}{\partial \dot{q}_\ell} = \frac{\partial T}{\partial \dot{q}_\ell}.$$

Thus,

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \frac{m}{\cancel{2}}\cancel{2}\dot{\rho} = m\dot{\rho},$$

$$p_\varphi = \frac{\partial T}{\partial \dot{\varphi}} = \frac{m}{\cancel{2}}\rho^2\cancel{2}\dot{\varphi} = m\rho^2\dot{\varphi},$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = \frac{m}{\cancel{2}}\cancel{2}\dot{z} = m\dot{z},$$

so that

$$\dot{\rho} = \frac{p_\rho}{m}, \quad \dot{\varphi} = \frac{p_\varphi}{m\rho^2}, \quad \dot{z} = \frac{p_z}{m}.$$

As a result, T expressed in terms of $\rho, \varphi, z, p_\rho, p_\varphi,$ and $p_z$ is

$$T = \frac{m}{2}\left(\frac{p_\rho^2}{m^2} + \rho^2 \frac{p_\varphi^2}{m^2\rho^4} + \frac{p_z^2}{m^2}\right)$$

$$= \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right).$$

In our case, $V = \frac{1}{2}K\rho^2$.

Thus,

$$\boxed{H = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + \frac{1}{2}K\rho^2}.$$

2. (6 pts.) Consider a normalized one-dimensional wave packet $\psi = \psi(x,\, t)$ under the influence of a force derived from the real potential $V = V(x,t)$. The mass of the corresponding classical particle is designated by $m$. Use the time-dependent Schrödinger equation to show that

$$\frac{d}{dt}\langle \hat{x} \rangle = \frac{1}{m}\langle \hat{p}_x \rangle, \tag{1}$$

where $\hat{x}$ and $\hat{p}_x$ are the coordinate and momentum operators, respectively, and

$$\langle \hat{F} \rangle = \int_{-\infty}^{+\infty} {}^*\hat{F}\psi\, dx$$

represents the expectation value of operator $\hat{F}$. Briefly comment on Eq. (1).
*Hints*: Use the equations

$$\int_{-\infty}^{+\infty} {}^*\frac{\partial^2 \phi}{\partial x^2}\, dx = \int_{-\infty}^{+\infty} \left(\frac{\partial^2 \psi^*}{\partial x^2}\right) \phi\, dx$$

and

$$\frac{\partial^2 fg}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2},$$

where $\psi$ and $\phi$ are functions that vanish at $x = \pm\infty$ and $f$ and $g$ are arbitrary functions.

*Solution*

$$\frac{d}{dt}\langle \hat{x} \rangle = \frac{d}{dt}\int_{-\infty}^{\infty} \psi^* \hat{x} \psi\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\psi^* x \psi)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\partial \psi^*}{\partial t}\right) x \psi\, dx + \int_{-\infty}^{\infty} \psi^* x \frac{\partial \psi}{\partial t}\, dx .$$

From the time dependent Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi ,$$

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{V}{i\hbar}\psi ,$$

$$\frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} - \frac{V}{i\hbar}\psi^* ;$$

where we used $V = V^*$ in the latter formula.

We obtain,

$$\frac{d}{dt}\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \left( -\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} - \frac{V}{i\hbar} \psi^* \right) x \psi \, dx$$

$$+ \int_{-\infty}^{\infty} \psi^* x \left( \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{V}{i\hbar} \psi \right) dx$$

$$= -\frac{i\hbar}{2m} \int_{-\infty}^{\infty} \frac{\partial^2 \psi^*}{\partial x^2} x \psi \, dx$$

$$\cancel{- \frac{1}{i\hbar} \int_{-\infty}^{\infty} V \psi^* x \psi \, dx}$$

$$+ \frac{i\hbar}{2m} \int_{-\infty}^{\infty} \psi^* x \frac{\partial^2 \psi}{\partial x^2} dx$$

$$\cancel{+ \frac{1}{i\hbar} \int_{-\infty}^{\infty} \psi^* x V \psi \, dx}$$

$$= -\frac{i\hbar}{2m} \left[ \int_{-\infty}^{\infty} \left( \frac{\partial^2 \psi^*}{\partial x^2} \right) (x \psi) \, dx - \int_{-\infty}^{\infty} \psi^* x \frac{\partial^2 \psi}{\partial x^2} dx \right].$$

The first of the two equations in the hints implies that

$$\int_{-\infty}^{\infty} \left( \frac{\partial^2 \psi^*}{\partial x^2} \right) (x \psi) \, dx = \int_{-\infty}^{\infty} \psi^* \frac{\partial^2}{\partial x^2} (x \psi) dx .$$

so that

$$\frac{d}{dt}\langle \hat{x} \rangle = -\frac{i\hbar}{2m}\int_{-\infty}^{\infty} \psi^* \left[ \frac{\partial^2}{\partial x^2}(x\psi) - x\frac{\partial^2 \psi}{\partial x^2} \right] dx .$$

Using the second equation in the hints, we obtain,

$$\frac{d}{dt}\langle \hat{x} \rangle = -\frac{i\hbar}{2m}\int_{-\infty}^{\infty} \psi^* \left[ 2\frac{\partial x}{\partial x}\frac{\partial \psi}{\partial x} + \cancel{x\frac{\partial^2 \psi}{\partial x^2}} - \cancel{x\frac{\partial^2 \psi}{\partial x^2}} \right] dx$$

$$= -\frac{i\hbar}{\cancel{2}m}\cancel{2}\int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x}\, dx$$

$$= \frac{1}{m}\int_{-\infty}^{\infty} \psi^* \left( \frac{\hbar}{i}\frac{\partial \psi}{\partial x} \right) dx$$

$$= \frac{1}{m}\int_{-\infty}^{\infty} \psi^* \hat{p}_x \psi \, dx = \frac{1}{m}\langle \hat{p}_x \rangle$$

Thus, indeed,

$$\boxed{\frac{d}{dt}\langle \hat{x} \rangle = \frac{1}{m}\langle \hat{p}_x \rangle} .$$

This result is an example of the Ehrenfest theorem. Classically,

$$\frac{dx}{dt} = \frac{1}{m}p_x .$$

In general, we can replace classical relations by quantum mechanical ones provided that classical variables are replaced by expectation values of the corresponding operators, so that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}$$

is classical mechanics becomes

$$\frac{d}{dt}\langle \hat{F} \rangle = \left\langle \frac{\partial \hat{F}}{\partial t} \right\rangle + \frac{1}{i\hbar}\langle [\hat{F}, \hat{H}] \rangle$$

in quantum mechanics.

In our case, the relation

$$\frac{dx}{dt} = \frac{1}{m} p_x ,$$

after replacing $x$ by $\langle \hat{x} \rangle$ and $p_x$ by $\langle \hat{p}_x \rangle$, becomes

$$\frac{d}{dt}\langle \hat{x} \rangle = \frac{1}{m}\langle \hat{p}_x \rangle .$$

3. (4 pts.) Let $\hat{H}$, $\hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$, $\hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z})$, and $\hat{V}$ designate the Hamiltonian, momentum, coordinate, and potential energy operators, respectively, for a single particle in three dimensions. Use the following commutator relationships:

$$[\hat{x}, \hat{H}] = \frac{i\hbar}{m} \hat{p}_x,$$

$$[\hat{p}_x, \hat{H}] = -i\hbar \frac{\partial \hat{V}}{\partial x},$$

and analogous formulas for $[\hat{y}, \hat{H}]$ and $[\hat{p}_y, \hat{H}]$ to show that

$$[\hat{L}_z, \hat{H}] = i\hbar \hat{D}_z,$$

where $\hat{L}_z$ and $\hat{D}_z$ are the operators representing the $z$ components of the angular momentum ($\mathbf{L} = \mathbf{r} \times \mathbf{p}$) and torque ($\mathbf{D} = \mathbf{r} \times \mathbf{F}$). Can you tell, without performing any additional calculations, what would the formulas for the Poisson brackets of $x$ and $H$, $p_x$ and $H$, and $L_z$ and $H$ be? If you can tell this, please write down these formulas.
*Auxiliary formula*: $[\hat{F}_1 \hat{F}_2, \hat{G}] = \hat{F}_1 [\hat{F}_2, \hat{G}] + [\hat{F}_1, \hat{G}] \hat{F}_2$.

*Solution*

$$[\hat{L}_z, \hat{H}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{H}] = [\hat{x}\hat{p}_y, \hat{H}] - [\hat{y}\hat{p}_x, \hat{H}]$$

$$\overset{\text{auxiliary formula}}{=} \hat{x}[\hat{p}_y, \hat{H}] + [\hat{x}, \hat{H}]\hat{p}_y - \hat{y}[\hat{p}_x, \hat{H}] - [\hat{y}, \hat{H}]\hat{p}_x$$

$$= -i\hbar\, \hat{x} \frac{\partial \hat{V}}{\partial y} + \cancel{\frac{i\hbar}{m} \hat{p}_x \hat{p}_y} + i\hbar\, \hat{y} \frac{\partial \hat{V}}{\partial x} - \cancel{\frac{i\hbar}{m} \hat{p}_y \hat{p}_x}$$

($\hat{p}_x, \hat{p}_y$ commute)

$$= i\hbar \left[ \hat{x} \left( -\frac{\partial \hat{V}}{\partial y} \right) - \hat{y} \left( -\frac{\partial \hat{V}}{\partial x} \right) \right]$$

In classical mechanics,

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y},$$

so that we can regard $\left(-\frac{\partial \hat{V}}{\partial x}\right)$ and $\left(-\frac{\partial \hat{V}}{\partial y}\right)$ as the x and y components of an operator representing force,

$$[\hat{L}_z, \hat{H}] = i\hbar\left(\hat{x}\hat{F}_y - \hat{y}\hat{F}_x\right) = i\hbar\left(\hat{\vec{r}} \times \hat{\vec{F}}\right)_z$$
$$= i\hbar \hat{D}_z \, .$$

Thus, indeed,

$$\boxed{[\hat{L}_z, \hat{H}] = i\hbar \hat{D}_z} \, .$$

In general, there is a relationship between commutators in quantum mechanics and Poisson brackets in classical mechanics,

$$\{F, G\} \Longrightarrow \frac{1}{i\hbar}[\hat{F}, \hat{G}] \, .$$

Thus,

$$\boxed{\begin{aligned} \{x, H\} &= \frac{p_x}{m} \, , \\ \{p_x, H\} &= -\frac{\partial V}{\partial x} \, , \\ \{L_z, H\} &= D_z \, . \end{aligned}}$$

4. (5 pts.) The ground and first-excited eigenstates of the one-dimensional harmonic oscillator are described by the following normalized wave functions:

$$u_0(x) = \sqrt{\frac{\alpha}{\pi^{\frac{1}{2}}}}\, e^{-\frac{1}{2}\alpha^2 x^2},$$

and

$$u_1(x) = \sqrt{\frac{2\alpha^3}{\pi^{\frac{1}{2}}}}\, x\, e^{-\frac{1}{2}\alpha^2 x^2},$$

respectively. Here, $\alpha = (\frac{m\omega}{\hbar})^{\frac{1}{2}}$, where $m$ is the oscillator's mass and $\omega$ is the angular frequency.

(a) Show that $u_0$ and $u_1$ are orthogonal. (1 pt.)

(b) The energy eigenvalue $E_0$ corresponding to the ground eigenstate $u_0$ is $\frac{1}{2}\hbar\omega$. Calculate the expectation value $\langle \hat{V} \rangle_0$ of the potential energy operator $\hat{V} = \frac{1}{2} m\omega^2 \hat{x}^2$ in the state described by $u_0$. Compare $\langle \hat{V} \rangle_0$ with $E_0$. (2 pts.)

(c) Based on the relationship between $\langle \hat{V} \rangle_0$ and $E_0$ obtained in part (b), determine the expectation value of the kinetic energy operator $\hat{T} = \frac{\hat{p}_x^2}{2m}$ in the state described by $u_0$. (2 pts.)

*Auxiliary formula*: $\int_{-\infty}^{+\infty} x^{2n} e^{-ax^2} dx = \frac{(2n)!}{2^{2n} n!}\sqrt{\frac{\pi}{a^{2n+1}}}$, $a > 0$, $n = 0, 1, 2, \ldots$ .

*Solution*

(a) $$(u_0, u_1) = \int_{-\infty}^{\infty} u_0^*(x)\, u_1(x)\, dx$$

$$= \sqrt{\frac{\alpha}{\pi^{\frac{1}{2}}}} \sqrt{\frac{2\alpha^3}{\pi^{\frac{1}{2}}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha^2 x^2}\, x\, e^{-\frac{1}{2}\alpha^2 x^2} dx$$

$$= \sqrt{\frac{2}{\pi}}\, \alpha^2 \int_{-\infty}^{\infty} x\, e^{-\alpha^2 x^2} dx = 0,$$

since $x e^{-\alpha^2 x^2}$ is an odd function of $x$. In general, if $f(x)$ is an odd function of $x$, i.e., $f(-x) = -f(x)$, $\int_{-\infty}^{\infty} f(x)\, dx = 0$.

Thus, $\int_{-\infty}^{\infty} x e^{-\alpha^2 x^2} dx = 0$ and, as a result,

$$\boxed{(u_0, u_1) = 0}.$$

(b) $$\langle \hat{V} \rangle_0 = \int_{-\infty}^{\infty} u_0^*(x) \hat{V} u_0(x) dx$$

$$= \frac{\alpha}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha^2 x^2} \left(\frac{1}{2} m\omega^2 \hat{x}^2\right) e^{-\frac{1}{2}\alpha^2 x^2} dx$$

$$= \frac{m\omega^2 \alpha}{2\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} dx .$$

From the auxiliary formula, applied to $n=1$ and $a = \alpha^2$, we obtain,

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} dx = \frac{2!}{2^2 1!} \sqrt{\frac{\pi}{\alpha^6}} = \frac{\sqrt{\pi}}{2\alpha^3}.$$

Thus,

$$\langle \hat{V} \rangle_0 = \frac{m\omega^2 \cancel{\alpha}}{2\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{2\alpha^{\cancel{3}\,2}} = \frac{m\omega^2}{4\alpha^2} .$$

We know that $\alpha = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}$, so that

$$\langle \hat{V} \rangle_0 = \frac{\cancel{m}\,\omega^{\cancel{2}}\hbar}{4\cancel{m}\cancel{\omega}} = \frac{1}{4}\hbar\omega .$$

$$\boxed{\langle \hat{V} \rangle_0 = \frac{1}{4}\hbar\omega = \frac{1}{2}E_0}.$$

(c) Since $\hat{H}u_0 = E_0 u_0$ and $\hat{H} = \hat{T} + \hat{V}$, we can write,

$$\langle \hat{T} + \hat{V} \rangle_0 = \langle \hat{H} \rangle_0 = \int_{-\infty}^{\infty} u_0^* \underbrace{\hat{H} u_0}_{E_0 u_0} dx$$

$$= \int_{-\infty}^{\infty} u_0^* E_0 u_0 \, dx = E_0 \underbrace{\int_{-\infty}^{\infty} u_0^* u_0 \, dx}_{1}$$

$$= E_0 .$$

On the other hand, $\langle \hat{T} + \hat{V} \rangle_0 = \langle \hat{T} \rangle_0 + \langle \hat{V} \rangle_0$. Thus,

$$E_0 = \langle \hat{T} + \hat{V} \rangle_0 = \langle \hat{T} \rangle_0 + \langle \hat{V} \rangle_0 ,$$

$$\langle \hat{T} \rangle_0 = E_0 - \langle \hat{V} \rangle_0 \overset{(b)}{=} \frac{1}{2}\hbar\omega - \frac{1}{4}\hbar\omega = \frac{1}{4}\hbar\omega,$$

$$\boxed{\langle \hat{T} \rangle_0 = \langle \hat{V} \rangle_0 = \frac{1}{2}E_0 = \frac{1}{4}\hbar\omega}.$$

5. (4 pts.) Operator $U$ is called *unitary* if (i) its domain is the entire Hilbert space ($D(U) = \mathcal{H}$) (ii) its range is the entire Hilbert space ($\Delta(U) = \mathcal{H}$), and (iii) for every $f$, $g \in \mathcal{H}$ we can write

$$(Uf, Ug) = (f, g). \tag{1}$$

The above definition of a unitary operator does not mention that unitary operators are also linear operators, and yet it can be shown that they are linear. In particular,

$$U(f+g) = Uf + Ug \tag{2}$$

for all $f$, $g \in \mathcal{H}$. Use the inner product expression $(U(f+g) - Uf - Ug, U(f+g) - Uf - Ug)$, the bilinearity and positive-definitiness of the inner product, and the property represented by Eq. (1) to show that Eq. (2) is indeed true.

*Solution*

$$(U(f+g) - Uf - Ug, U(f+g) - Uf - Ug)$$

$$\overset{\text{bilinearity of inner product}}{=} (U(f+g), U(f+g)) - (U(f+g), Uf)$$
$$- (U(f+g), Ug) - (Uf, U(f+g))$$
$$+ (Uf, Uf) + (Uf, Ug)$$
$$- (Ug, U(f+g)) + (Ug, Uf) + (Ug, Ug)$$

$$\overset{\text{Eq.(1)}}{=} (f+g, f+g) - (f+g, f) - (f+g, g)$$
$$- (f, f+g) + (f, f) + (f, g)$$
$$- (g, f+g) + (g, f) + (g, g)$$

$$\overset{\text{bilinearity of inner product}}{=} (f+g, \cancel{f}+\cancel{g}-\cancel{f}-\cancel{g}) - (f, \cancel{f}+\cancel{g}-\cancel{f}-\cancel{g})$$
$$- (g, \cancel{f}+\cancel{g}-\cancel{f}-\cancel{g})$$

$$= (f+g,\Theta) - (f,\Theta) - (g,\Theta) \overset{\text{bilinearity}}{\underset{\text{of inner product}}{=}} (\cancel{f}+\cancel{g}-\cancel{f}-\cancel{g},\Theta) = (\Theta,\Theta) = 0,$$

where $\Theta$ is a zero vector.

Thus,

$$\big(U(f+g) - Uf - Ug,\, U(f+g) - Uf - Ug\big) = 0.$$

The positive-definitiness of the inner product requires that $(h,h) \geq 0$ and $(h,h) = 0$ implies that $h = \Theta$.

Thus,

$$U(f+g) - Uf - Ug = \Theta,$$

i.e.,

$$U(f+g) = Uf + Ug,$$

which is Eq. (2).