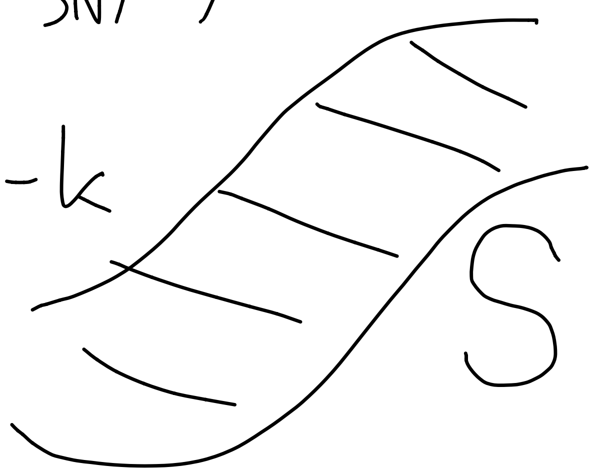
$$\begin{cases} f_1(x_1, \ldots, x_{3N}, t) = 0 \\ \ldots \\ f_k(x_1, \ldots, x_{3N}, t) = 0 \end{cases}$$

$$f = 3N - k$$

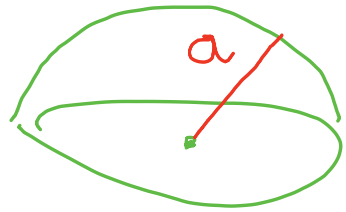


$N = 1$

$$x^2 + y^2 + z^2 - a^2 = 0$$

$$z = \pm\sqrt{a^2 - x^2 - y^2}$$

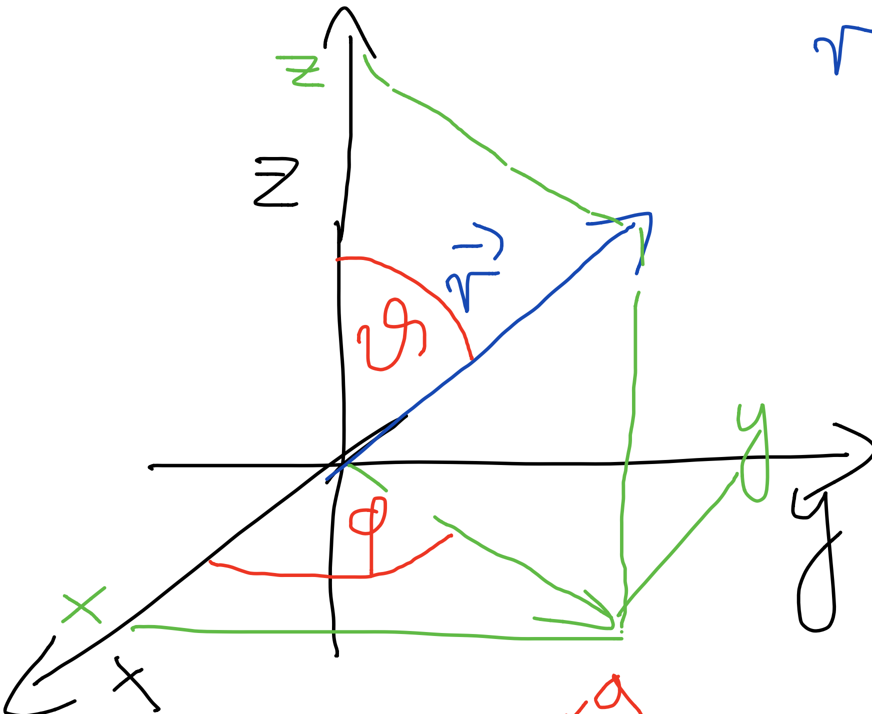
$$z = z(x, y)$$



$\vec{r} = (x, y, z)$

r, ϑ, φ

$\vartheta = \theta$



$$x = \overset{a}{\cancel{r}} \sin\vartheta \cos\varphi$$

$$y = \overset{a}{\cancel{r}} \sin\vartheta \sin\varphi$$

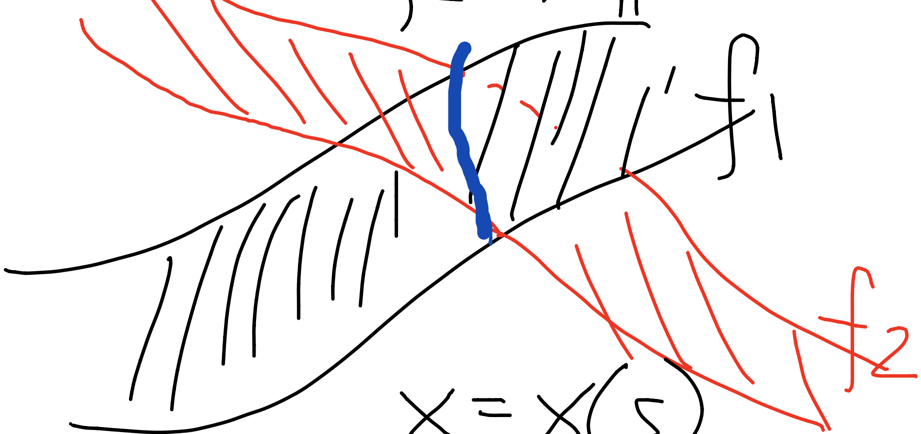
$$z = \overset{a}{\cancel{r}} \cos\vartheta$$

$r = a$ (fixed) $\quad \vartheta, \varphi$

$N=1$

$$f_1(x,y,z)=0$$

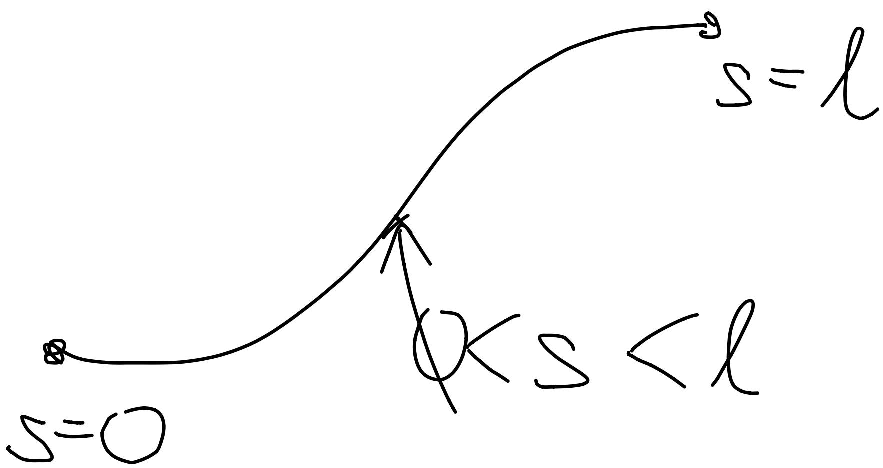
$$f_2(x,y,z)=0$$



$$x=x(s)$$

$$y=y(s)$$

$$z=z(s)$$



$q_1, \dots, q_f \quad , \quad f = 3N - k$

$$\begin{cases} x_1 = x_1(q_1, \dots, q_f, t) \\ \dots \\ x_{3N} = x_{3N}(q_1, \dots, q_f, t) \end{cases}$$

$\boxed{\text{generalized coordinates}}$

$q_1 = \vartheta$

$q_2 = \varphi$

$$x_j = x_j(\overbrace{q_1, \dots, q_f}^{q}, t)$$

$j = 1, \dots, 3N$

$$x_j = x_j(q, t)$$

$$x = x(q, t)$$

✓ $$f_p(x_1(q,t), \dots, x_{3N}(q,t)) \equiv 0$$

$$p = 1, \dots, k$$

$$\boxed{\frac{\partial f_p}{\partial q_l} \equiv 0}$$

$$\delta q = (\delta q_1, \dots, \delta q_f)$$

✓ $$\delta x_j = \sum_{l=1}^{f} \frac{\partial x_j}{\partial q_l} \delta q_l \quad \cancel{\left(+ \frac{\partial x_j}{\partial t} \delta t\right)}$$

$$j = 1, \dots, 3N$$

$$0 = \sum_{j=1}^{3N} \frac{\partial f_p}{\partial x_j} \delta x_j$$

$$= \sum_{j=1}^{3N} \frac{\partial f_p}{\partial x_j} \sum_{l=1}^{f} \frac{\partial x_j}{\partial q_l} \delta q_l$$

$$= \sum_{l=1}^{f} \underbrace{\left(\sum_{j=1}^{3N} \frac{\partial f_p}{\partial x_j} \frac{\partial x_j}{\partial q_l} \right)}_{= \frac{\partial f_p}{\partial q_l}} \delta q_l$$

$$= \sum_{l=1}^{f} \underbrace{\frac{\partial f_p}{\partial q_l}}_{=0} \delta q_l = 0$$

$\checkmark$ $$Q_l = \sum_{j=1}^{3N} X_j \frac{\partial x_j}{\partial q_l}$$

$$l = 1, \ldots, f$$

$$x, y, z \longrightarrow r, \vartheta, \varphi$$

$$F_x, F_y, F_z \longrightarrow Q_r, Q_\vartheta, Q_\varphi$$

$$Q_r = F_x \frac{\partial x}{\partial r} + F_y \frac{\partial y}{\partial r} + F_z \frac{\partial z}{\partial r}$$

$$\sum_{j=1}^{3N}\left(X_j - m_j \ddot{x}_j\right)\delta x_j = 0$$

✓ $$\sum_{j=1}^{3N} X_j\,\delta x_j = ?$$

✓ $$\sum_{j=1}^{3N} m_j \ddot{x}_j\,\delta x_j = ?$$

✓ $$\sum_j X_j\,\delta x_j = \sum_j X_j \sum_l \frac{\partial x_j}{\partial q_l}\delta q_l$$

$$= \sum_l \left(\sum_j X_j \frac{\partial x_j}{\partial q_l}\right)\delta q_l$$

$$= \sum_l Q_l \, \delta q_l$$

$$\boxed{\begin{gathered} \frac{\partial \dot{x}_j}{\partial \dot{q}_l} = \frac{\partial x_j}{\partial q_l} \\[1em] \frac{\partial \dot{x}_j}{\partial q_l} = \frac{d}{dt} \frac{\partial x_j}{\partial q_l} \end{gathered}}$$

$$\sum_{j=1}^{3N} m_j \ddot{x}_j \delta x_j$$

$$= \sum_{j=1}^{3N} m_j \ddot{x}_j \sum_{l=1}^{f} \frac{\partial x_j}{\partial q_l} \delta q_l$$

$$= \sum_l \underbrace{\left(\boxed{\sum_j m_j \ddot{x}_j \frac{\partial x_j}{\partial q_l}} \right)}\delta q_l$$

$$\boxed{\sum_j m_j \ddot{x}_j \frac{\partial x_j}{\partial q_l} =}$$

$$= \sum_j \left\{ \frac{d}{dt}\left[m_j \dot{x}_j \frac{\partial x_j}{\partial q_l} \right] \right.$$

$$- \; m_j \dot{x}_j \frac{d}{dt} \frac{\partial x_j}{\partial q_l} \Big\}$$

$$\frac{d}{dt}\left(m_j \dot{x}_j \frac{\partial x_j}{\partial q_l}\right) = m_j \ddot{x}_j \frac{\partial x_j}{\partial q_l} + m_j \dot{x}_j \frac{d}{dt}\frac{\partial x_j}{\partial q_l}$$

$$\sum_j m_j \ddot{x}_j \frac{\partial x_j}{\partial q_l} = \sum_j \frac{d}{dt}\left[m_j \dot{x}_j \frac{\partial x_j}{\partial q_l}\right] - \sum_j m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial q_l}$$

$$= \sum_j \frac{d}{dt}\left[\frac{1}{2} m_j \frac{\partial}{\partial \dot{q}_l}(\dot{x}_j)^2\right]$$

$$- \sum_j \frac{1}{2} m_j \frac{\partial}{\partial q_l}(\dot{x}_j)^2$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_l}\left(\sum_j \frac{1}{2} m_j \dot{x}_j^2\right)$$

$$- \frac{\partial}{\partial q_l}\left(\sum_j \frac{1}{2} m_j \dot{x}_j^2\right)$$

T

T

$$\sum_j m_j \ddot{x}_j \frac{\partial x_j}{\partial q_l} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial T}{\partial q_l}$$

$$\sum_j m_j \ddot{x}_j \,\delta x_j = \sum_l \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial T}{\partial q_l}\right)\delta q_l$$

$$\sum_j X_j \,\delta x_j = \sum_l Q_l \,\delta q_l$$

$$\sum_j \left(X_j - m_j \ddot{x}_j\right)\delta x_j$$

$$= \sum_l \underbrace{\left(Q_l - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} + \frac{\partial T}{\partial q_l}\right)}_{0} \times \delta q_l = 0$$

$$\boxed{Q_l = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial T}{\partial q_l} \qquad l = 1, \dots, f}$$

Lagrange's equations of the 2nd kind

2nd order ODEs

$q_{l0} = q_l(t_0)$; $\dot{q}_{l0} = \dot{q}_l(t_0)$

$$x_j = x_j(q_1, \ldots, q_f, t)$$

$$\dot{x}_j = \sum_{k=1}^{f} \frac{\partial x_j}{\partial q_k} \dot{q}_k + \frac{\partial x_j}{\partial t}$$

$(q, \dot{q}, t)$

δ_{kl}

$$\frac{\partial \dot{x}_j}{\partial \dot{q}_l} = \sum_{k=1}^{f} \frac{\partial x_j}{\partial q_k} \left(\frac{\partial \dot{q}_k}{\partial \dot{q}_l} \right)$$

$$\delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$$

$$= \frac{\partial x_j}{\partial q_l} \quad \checkmark$$

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_l} \right) = \sum_{k=1}^{f} \frac{\partial x_j^2}{\partial q_k \partial q_l} \dot{q}_k$$

$$+ \frac{\partial x_j^2}{\partial t \partial q_l}$$

$$\frac{\partial \dot{x}_j}{\partial q_l} = \sum_{k=1}^{f} \frac{\partial^2 x_j}{\partial q_l \partial q_k} \dot{q}_k + \frac{\partial^2 x_j}{\partial q_l \partial t}$$

$$X_j = -\frac{\partial V}{\partial x_j}$$

$$j = 1, \ldots, 3N$$

$$V = V(x(q, t), t)$$

$$Q_l = \sum_j X_j \frac{\partial x_j}{\partial q_l}$$

$$= -\sum_j \frac{\partial V}{\partial x_j} \frac{\partial x_j}{\partial q_l} = \frac{\partial V}{\partial q_l}$$

$$\boxed{Q_l = -\frac{\partial V}{\partial q_l}}$$

$$\frac{\partial V}{\partial \dot{q}_l} = 0$$

$$Q_l = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial T}{\partial q_l}$$

$$= -\frac{\partial V}{\partial q_l}\left(+\frac{d}{dt}\frac{\partial V}{\partial \dot{q}_l}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_l}(T-V)$$

$$= \frac{\partial}{\partial q_l}(T-V)$$

$$L = T - V$$

Lagrange equation

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} = 0 \qquad l = 1, \ldots, f}$$

$$f = 3N - k$$

$$L(q, \dot{q}, t) = T(\dot{x}(q, \dot{q}, t)) - V(x(q, t), t)$$

$$Q_l = -\frac{\partial U}{\partial q_l} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_l}$$

$$U \equiv U(q, \dot{q}, t)$$

generalized
(or velocity dependent)
potential

(In Cartesian coordinates,

$$X_j = -\frac{\partial U}{\partial x_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}_j}$$
)

If $\frac{\partial U}{\partial \dot{x}_j} = 0 \Rightarrow U$ becomes V

$$Q_l = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l} - \frac{\partial T}{\partial q_l}$$

$$= -\frac{\partial U}{\partial q_l} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_l}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} = 0$$

$$L = T - U$$

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\left(\vec{v} \times \vec{B}\right)\right)$$

$$\dot{\vec{r}}$$

$$\rho = \rho(x, y, z, t)$$

$$\vec{j} = \vec{j}(x, y, z, t)$$

$\rho \to$ charge density

$\vec{j} \to$ current density

✓ $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ Gauss law

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

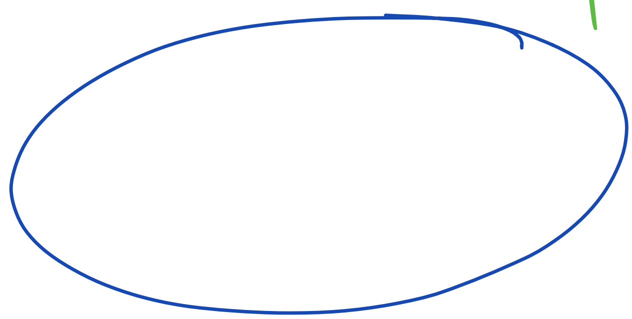
$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\vec{E} = \vec{E}(x, y, z, t)$$

$$\vec{B} = \vec{B}(x, y, z, t)$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

(no magnetic monopoles)



$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

(Faraday's law)

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

If $\frac{\partial \vec{E}}{\partial t} = 0$

$\Rightarrow$ Ampere's law

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\Downarrow$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

magnetic
vector
potential

$$\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{A} \right) = 0$$

$$= \vec{\nabla} \times \underbrace{\left[\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right]}_{} = 0$$

If $\vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{F} = \vec{\nabla} f$

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\varphi$$

$$\varphi = \varphi(x, y, z, t)$$

electric potential

$$\boxed{\begin{aligned} \vec{E} &= -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}}$$

$$\rightarrow \quad U = q\varphi - \frac{q}{c}\underbrace{\vec{A}\cdot\vec{v}}_{A_x\dot{x} + A_y\dot{y} + A_z\dot{z}}$$

$$\boxed{F_x = \frac{d}{dt}\frac{\partial U}{\partial \dot{x}} - \frac{\partial U}{\partial x}}$$

$$\varphi = \varphi(x, y, z, t)$$

$$\vec{A} = \vec{A}(x, y, z, t)$$

e.g., $A_x = A_x(x, y, z, t)$

$$\mathcal{L} = T - U$$

$$= T - q\varphi + \frac{q}{c}\vec{A}\cdot\vec{v}$$