HARMONIC OSCILLATOR (REVISITED)

In classical mechanics,

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2},$$

$$\{x, p\} = 1$$

In quantum mechanics,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2},$$

$$[\hat{x}, \hat{p}] = i\hbar \mathbb{1},$$

$$\hat{H}|u_\mu\rangle = E_\mu |u_\mu\rangle$$

From now on, we will drop the "^" representing operators.

Preparatory steps

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)$$

$$= \frac{1}{2}\hbar\omega\left(P^2 + Q^2\right) \quad ✓$$

$$P = \frac{p}{(\hbar\omega m)^{\frac{1}{2}}}, \quad Q = \underbrace{\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}}_{\alpha} x$$

$$Q = \alpha x = \xi$$

$$x, p \longrightarrow Q, P$$

$$[Q, P] = \left(\frac{\cancel{m\omega}}{\hbar}\right)^{\frac{1}{2}} \frac{1}{(\hbar\cancel{\omega m})^{\frac{1}{2}}}$$

$$\times [x, p] = \frac{1}{\hbar} i\hbar = i$$

$$H = \hbar\omega\mathcal{H}$$

$$\mathcal{H} = \frac{1}{2}\left(P^2 + Q^2\right), \quad [Q, P] = i$$

$$x, p \longrightarrow Q, P \longrightarrow a, a^\dagger$$

$$a = \frac{1}{\sqrt{2}}(Q + iP) \quad \leftarrow \text{annihilation operator}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) \quad \leftarrow \text{creation operator}$$

$$[a, a^\dagger] = \frac{1}{2}[Q + iP, Q - iP]$$

$$= \frac{1}{2}\Big(\cancelto{0}{[Q,Q]} - i\overbrace{[Q,P]}^{i} + i\underbrace{[P,Q]}_{-i} + \cancelto{0}{[P,P]}\Big)$$

$$= \frac{1}{2}(1 + 1) = 1$$

$$[a, a^\dagger] = 1 \quad \checkmark$$

$$a + a^\dagger = \frac{2}{\sqrt{2}} Q = \sqrt{2}\, Q$$

$$a - a^\dagger = \frac{2i}{\sqrt{2}} P = i\sqrt{2}\, P$$

$$Q = \frac{1}{\sqrt{2}}(a + a^\dagger)$$

$$P = \frac{1}{i\sqrt{2}}(a - a^\dagger)$$

$$\mathcal{H} = \frac{1}{2}\left(Q^2 + P^2\right)$$

$$= \frac{1}{2}\left\{ \frac{1}{2}(a + a^\dagger)^2 - \frac{1}{2}(a - a^\dagger)^2 \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{2}\left(\cancel{a^2} + \cancel{a^{\dagger 2}} + \underline{a a^\dagger} + \underline{a^\dagger a}\right) - \frac{1}{2}\left(\cancel{a^2} + \cancel{a^{\dagger 2}} - \underline{a a^\dagger} - \underline{a^\dagger a}\right) \right\}$$

$$= \frac{1}{2}\left(a a^\dagger + a^\dagger a\right)$$

$$\mathcal{H} = \frac{1}{2}\left(aa^\dagger + \underbrace{a^\dagger a}_{N}\right)$$

$$N = a^\dagger a \longleftarrow \text{number operator}$$

$$[a, a^\dagger] = 1$$

$$aa^\dagger - a^\dagger a = 1$$

$$aa^\dagger = 1 + a^\dagger a = 1 + N$$

$$\mathcal{H} = \frac{1}{2}(1 + N + N)$$

$$= N + \frac{1}{2}$$

$$H = \hbar\omega\mathcal{H}$$

$$\mathcal{H} = N + \frac{1}{2}$$

$$N = a^\dagger a, \quad [a, a^\dagger] = 1$$

N is self adjoint (hermitian)

$$N^\dagger = (a^\dagger a)^\dagger = a^\dagger a^{\dagger\dagger} = a^\dagger a$$

$$[(AB)^\dagger = B^\dagger A^\dagger] \qquad = N$$

Useful relations:

$$[N, a] = -a \quad \checkmark$$

$$[N, a^\dagger] = a^\dagger \quad \checkmark$$

Proof:

$$[N, a] = Na - aN$$

$$= a^\dagger a a - a a^\dagger a$$

$$= (a^\dagger a - a a^\dagger) a$$

$$= -\underbrace{[a, a^\dagger]}_{1} a = -a$$

$$[N, a^\dagger] = Na^\dagger - a^\dagger N$$
$$= N^\dagger a^\dagger - a^\dagger N^\dagger = (aN)^\dagger$$
$$- (Na)^\dagger = (aN - Na)^\dagger$$
$$= -[N, a]^\dagger = a^\dagger$$

Now, we proceed toward solving

$$H|u_\mu\rangle = E_\mu |u_\mu\rangle$$

$$N|u_\mu\rangle = a_\mu |u_\mu\rangle \quad ✓$$

$$H = \hbar\omega\left(N + \frac{1}{2}\right)$$

$$E_\mu = \hbar\omega\left(a_\mu + \frac{1}{2}\right) \quad ✓$$

$$|u_\mu\rangle \longrightarrow |\mu\rangle$$

$$\alpha_\mu \longrightarrow \mu \quad \checkmark$$

$$N|\mu\rangle = \mu|\mu\rangle$$

[i.e., $E_\mu = \hbar\omega(\mu + \frac{1}{2})$]

Theorem:

Let $|\mu\rangle$ be the eigenstates of N, $N|\mu\rangle = \mu|\mu\rangle$.

We can prove the following:

(i) $\mu \geq 0$. $\checkmark$

(ii) $\mu = 0$ implies that

$\checkmark$ $a|\mu\rangle = \Theta$ (we will be writing as 0)

Otherwise, for $\mu > 0$,

✓ $\| a|\mu\rangle \| > 0$ and

✓ $\| a|\mu\rangle \|^2 = \mu \| |\mu\rangle \|^2$.

(iii) $\| a^\dagger|\mu\rangle \| > 0$ ✓

and

✓ $\| a^\dagger|\mu\rangle \|^2 = (\mu+1) \| |\mu\rangle \|^2$.

(iv)

✓ $N[a|\mu\rangle] = (\mu-1)[a|\mu\rangle]$

✓ $N[a^\dagger|\mu\rangle] = (\mu+1)[a^\dagger|\mu\rangle]$

Proof:

(i) + (ii):

$$(a|\mu\rangle, a|\mu\rangle) \qquad N|\mu\rangle = \mu|\mu\rangle$$

$$\underbrace{\| a|\mu\rangle \|^2}_{\geq 0} = \langle \mu | \underbrace{a^\dagger a}_{N} | \mu \rangle$$

$$= \mu \langle \mu | \mu \rangle = \underbrace{\mu}_{\geq 0} \underbrace{\| |\mu\rangle \|^2}_{> 0}$$

$$\mu \geq 0$$

If $\mu = 0$,

$$\| a|\mu\rangle \|^2 = 0,$$

$$\| a|\mu\rangle \| = 0$$

$$a|\mu\rangle = 0_{\mathcal{H}}$$

If $\mu > 0$,

$$\| a|\mu\rangle \|^2 > 0,$$

$$\| a|\mu\rangle \| > 0.$$

(iii):

$$\| a^\dagger|\mu\rangle \|^2 \overset{(a^\dagger|\mu\rangle, a^\dagger|\mu\rangle)}{=} \langle \mu | a^{\dagger\dagger} a^\dagger | \mu \rangle$$

$$= \langle \mu | a a^\dagger | \mu \rangle$$

$$a a^\dagger - a^\dagger a = 1$$

$$a a^\dagger = N + 1$$

$$= \langle \mu | N + 1 | \mu \rangle$$

$$N|\mu\rangle = \mu|\mu\rangle$$

$$= (\mu+1) \langle \mu | \mu \rangle = (\mu+1) \| |\mu\rangle \|^2$$

$$\| a^\dagger |\mu\rangle \|^2 = \underset{\substack{\vee \\ 0}}{(\mu+1)} \, \| \underset{\substack{\vee \\ 0}}{|\mu\rangle} \|^2$$

$$> 0.$$

$$\| a^\dagger |\mu\rangle \| > 0 .$$

(iv):

$$N a|\mu\rangle = (aN - a)|\mu\rangle$$

$$[N, a] = -a, \quad Na - aN = -a \ \checkmark$$

$$[N, a^\dagger] = a^\dagger, \quad Na^\dagger - a^\dagger N = a^\dagger \ \checkmark$$

$$N a|\mu\rangle = a(N-1)|\mu\rangle$$

$$\mu|\mu\rangle$$

$$= a(\mu - 1)|\mu\rangle$$

$$= (\mu - 1)\, a|\mu\rangle \,.$$

$$N a^{\dagger}|\mu\rangle = (a^{\dagger}N + a^{\dagger})|\mu\rangle$$

$$= a^{\dagger}(N+1)|\mu\rangle$$

$$\mu|\mu\rangle$$

$$= a^{\dagger}(\mu+1)|\mu\rangle$$

$$= (\mu+1)\, a^{\dagger}|\mu\rangle$$

$$\therefore$$

We will now use the above theorem.

According to the theorem,

if $\mu = 0$, then

$$a|\mu\rangle \equiv a|0\rangle = 0$$

$$N|0\rangle = 0|0\rangle = 0$$

$$a^\dagger / \; a|0\rangle = 0$$

$$a^\dagger a|0\rangle = 0$$

$$N|0\rangle = 0$$

$$E_\mu = \hbar\omega\left(\mu + \frac{1}{2}\right) \checkmark$$

$$E_0 = \frac{1}{2}\hbar\omega \; (>0)$$

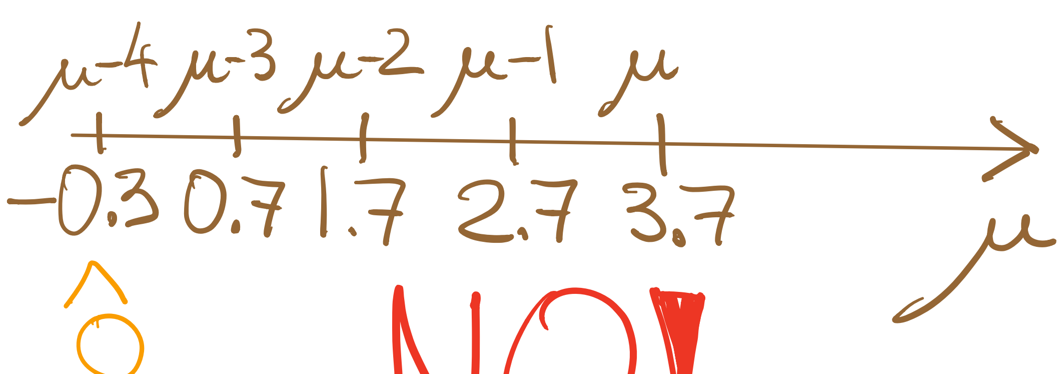
$|0\rangle$ is the ground state

To obtain the remaining eigenvalues and eigenstates, we will consider

$$|\mu\rangle,\ a|\mu\rangle,\ a^2|\mu\rangle, \dots, a^k|\mu\rangle$$

eigenvalue of N	μ	$\mu-1$	$\mu-2$	$\mu-k$

Can μ be a noninteger number? $(\mu \geq 0)$



NO!

μ MUST BE A NONNEGATIVE INTEGER. ✓

In general,

$$a^k|\mu\rangle, \quad k>\mu$$

would be an eigenstate of N with an eigenvalue $(\mu-k)<0$ if μ was not a nonnegative integer.

From now on,

$$\mu \longrightarrow \underset{\text{integer}}{n}, \quad n\geq 0$$

$$N|n\rangle = n|n\rangle,$$
$$n=0,1,2,\dots$$

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right),$$

$$H|n\rangle = E_n|n\rangle.$$

$$a^{n+1}|n\rangle = 0$$

$$a^{n+k}|n\rangle = 0 \quad \text{for} \quad k \geq 1.$$

Let us now talk about the eigenstates $|n\rangle$.

Let us consider

$$|0\rangle,\ a^\dagger|0\rangle, (a^\dagger)^2|0\rangle, \ldots (a^\dagger)^n|0\rangle, \ldots$$

eigen-value of N: $\quad 0 \quad 1 \quad 2 \quad n$

$$|n\rangle \sim (a^\dagger)^n|0\rangle$$

Is the set

$$\mathcal{B} = \{(a^\dagger)^n|0\rangle\} \checkmark$$

complete?

We know that $\mathcal{B}$ is orthogonal (because $(a^\dagger)^n|0\rangle$ are eigenstates of N).

To prove completeness, we must show that

$$\hat{F}\mathcal{B} \subseteq \mathcal{B} \quad \checkmark$$

$$\hat{F} = F(\hat{x}, \hat{p}, t)$$

Proof:

$$\hat{x} \equiv x = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}} Q$$

$$x = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}} (a + a^{\dagger})$$

$$\hat{p} \equiv p = (\hbar\omega m)^{\frac{1}{2}} P$$

$$= -i\left(\frac{\hbar\omega m}{2}\right)^{\frac{1}{2}} (a - a^{\dagger})$$

$\hat{F}$ is a function of a and $a^{\dagger}$.

We must prove that $a\mathcal{B} \subseteq \mathcal{B}$

and $a^{\dagger}\mathcal{B} \subseteq \mathcal{B}$.

$n=0$ obvious ($a|0\rangle = \theta \in \mathcal{B}$)

$$\underbrace{a(a^{\dagger})^n|0\rangle}_{\in \mathcal{B}} \overset{n>0}{=} \underbrace{aa^{\dagger}}_{a^{\dagger}a+1=N+1}(a^{\dagger})^{n-1}|0\rangle$$

$$= (N+1)\underbrace{(a^{\dagger})^{n-1}|0\rangle}_{\text{has an eigenvalue of } (n-1)}$$

$$= (n-1+1)(a^{\dagger})^{n-1}|0\rangle$$

$$= n(a^{\dagger})^{n-1}|0\rangle \in \mathcal{B}$$

$$a^{\dagger}\underbrace{(a^{\dagger})^n|0\rangle}_{\in \mathcal{B}} = (a^{\dagger})^{n+1}|0\rangle \in \mathcal{B}$$

$a\mathcal{B},\ a^\dagger\mathcal{B} \subset \mathcal{B}$

$\mathcal{B}$ is a complete set (space)!

Conclusion:

All eigenstates of H or N can be obtained by acting with $a^\dagger$ and its powers on $|0\rangle$,

$$c_n (a^\dagger)^n |0\rangle \equiv \underbrace{|n\rangle}_{\text{normalized}}$$

Note: there are no degeneracies,

$$H|n\rangle = E_n|n\rangle$$

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right),$$

$$n = 0, 1, \ldots$$

Let us determine c_n:

From our theorem (points (ii) & (iii)):

$$a|n\rangle = c|n-1\rangle \checkmark$$

$$\| a|n\rangle \|^2 = c^2 \| |n-1\rangle \|^2$$

$$= c^2$$

$$\rightarrow n \| |n\rangle \|^2 = n$$

$$c^2 = n$$

$$c = n^{\frac{1}{2}}$$

$$a^\dagger|n\rangle = \alpha|n+1\rangle \quad \checkmark$$

$$\| a^\dagger|n\rangle \|^2 = \alpha^2 \| |n+1\rangle \|^2$$

$$= \alpha^2$$

$$\rightarrow (n+1) \| |n\rangle \|^2 = n+1$$

$$\alpha^2 = n+1$$

$$\alpha = (n+1)^{\frac{1}{2}}$$

$$a|n\rangle = n^{\frac{1}{2}}|n-1\rangle$$

$$a^\dagger|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$$

$$|n+1\rangle = (n+1)^{-\frac{1}{2}} a^\dagger|n\rangle$$

$$|n\rangle = n^{-\frac{1}{2}} a^\dagger|n-1\rangle$$

$$|n\rangle = n^{-\frac{1}{2}} a^{\dagger} |n-1\rangle$$

$$= n^{-\frac{1}{2}} (n-1)^{-\frac{1}{2}} (a^{\dagger})^2 |n-2\rangle$$

$$= \dots = [n(n-1)\dots 1]^{-\frac{1}{2}}$$

$$\times (a^{\dagger})^n |0\rangle$$

$$= (n!)^{-\frac{1}{2}} (a^{\dagger})^n |0\rangle$$

$$c_n = (n!)^{-\frac{1}{2}}$$

$$|n\rangle = (n!)^{-\frac{1}{2}} (a^{\dagger})^n |0\rangle$$

normalized

normalized ground state

$$N|n\rangle = n|n\rangle$$

$$\langle n|n'\rangle = \delta_{nn'}.$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$(n = 0, 1, \dots)$$