SCHRÖDINGER'S PICTURE

$$|\varphi_S(t)\rangle, \quad \hat{F}_S = F(\hat{q}_S, \hat{p}_S, t)$$

$$\boxed{i\hbar \frac{d}{dt}|\varphi_S(t)\rangle = \hat{H}|\varphi_S(t)\rangle}$$

$$\boxed{\begin{aligned}\frac{d}{dt}\langle\varphi_S(t)|\hat{F}_S|\psi_S(t)\rangle &= \langle\varphi_S(t)|\frac{\partial \hat{F}_S}{\partial t}|\psi_S(t)\rangle \\ &+ \frac{1}{i\hbar}\langle\varphi_S(t)|[\hat{F}_S, \hat{H}]|\psi_S(t)\rangle\end{aligned}}$$

HEISENBERG'S PICTURE

$$|\varphi_S(t)\rangle = \hat{U}(t)\,|\varphi_S(\underset{t=0}{0})\rangle \checkmark$$

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} \qquad \checkmark$$

$$\hat{U}^\dagger = \hat{U}^{-1}$$

$$\|\varphi_S(t)\| = \|\varphi_S(0)\|$$

$$\frac{d}{dt}|\varphi_S(t)\rangle = \frac{d}{dt}\, e^{-i\hat{H}t/\hbar}\,|\varphi_S(0)\rangle$$

$$= \frac{d}{dt}\sum_{n=0}^{\infty}\frac{(-i\hat{H}t/\hbar)^n}{n!}\,|\varphi_S(0)\rangle$$

$$= \sum_{n=1}^{\infty} \frac{\cancel{n}\,(-i\hat{H}t/\hbar)^{n-1}}{\cancel{n!}\;(n-1)!} \left(\frac{-i\hat{H}}{\hbar}\right) \times |\varphi_s(0)\rangle$$

$$\overset{n-1=k}{=} -\frac{i}{\hbar}\hat{H} \sum_{k=0}^{\infty} \frac{(-i\hat{H}t/\hbar)^{k}}{k!} |\varphi_s(0)\rangle$$

$$= \frac{1}{i\hbar}\hat{H}\, \underbrace{e^{-i\hat{H}t/\hbar} |\varphi_s(0)\rangle}_{|\varphi_s(t)\rangle}$$

$$\frac{d}{dt}|\varphi_s(t)\rangle = \frac{1}{i\hbar}\hat{H}|\varphi_s(t)\rangle$$

$$i\hbar\frac{d}{dt}|\varphi_s(t)\rangle = \hat{H}|\varphi_s(t)\rangle$$

$$\therefore$$

$$\frac{d}{dt}\langle \varphi_S(t)|\hat{F}_S|\psi_S(t)\rangle =$$

$$|\psi_S(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi_S(0)\rangle$$

$$|\varphi_S(t)\rangle = e^{-i\hat{H}t/\hbar}|\varphi_S(0)\rangle$$

$$\langle \varphi_S(t)| = \langle \varphi_S(0)| e^{i\hat{H}t/\hbar}$$

$$= \frac{d}{dt}\langle \varphi_S(0)| e^{i\hat{H}t/\hbar}\hat{F}_S e^{-i\hat{H}t/\hbar}|\psi_S(0)\rangle$$

$$= \langle \varphi_S(0)| e^{i\hat{H}t/\hbar}\frac{\partial \hat{F}_S}{\partial t} e^{-i\hat{H}t/\hbar}|\psi_S(0)\rangle$$

$$+ \frac{1}{i\hbar}\langle \varphi_S(0)| e^{i\hat{H}t/\hbar}[\hat{F}_S, \hat{H}] e^{-i\hat{H}t/\hbar}|\psi_S(0)\rangle$$

$$e^{i\hat{H}t/\hbar}[\hat{F}_S, \hat{H}]e^{-i\hat{H}t/\hbar}$$

$$= e^{i\hat{H}t/\hbar}(\hat{F}_S\hat{H} - \hat{H}\hat{F}_S)e^{-i\hat{H}t/\hbar}$$

$$[e^{\pm i\hat{H}t/\hbar}, \hat{H}] = 0$$

$$= (e^{i\hat{H}t/\hbar}\hat{F}_S e^{-i\hat{H}t/\hbar})\hat{H}$$

$$- \hat{H}(e^{i\hat{H}t/\hbar}\hat{F}_S e^{-i\hat{H}t/\hbar})$$

$$= [e^{i\hat{H}t/\hbar}\hat{F}_S e^{-i\hat{H}t/\hbar}, \hat{H}]$$

Remark:

$$\hat{A} \longrightarrow \hat{S}^{-1}\hat{A}\hat{S},$$

$$\hat{S} = \hat{U}(t) = e^{-i\hat{H}t/\hbar}$$

$$\hat{A} \longrightarrow \hat{U}^\dagger \hat{A} \hat{U}$$

$$\frac{d}{dt}\langle \varphi_S(0)| e^{i\hat{H}t/\hbar} \hat{F}_S e^{-i\hat{H}t/\hbar} |\varphi_S(0)\rangle$$

$$= \langle \varphi_S(0)| e^{i\hat{H}t/\hbar} \frac{\partial \hat{F}_S}{\partial t} e^{-i\hat{H}t/\hbar} |\varphi_S(0)\rangle$$

$$+ \frac{1}{i\hbar}\langle \varphi_S(0)| \left[e^{i\hat{H}t/\hbar} \hat{F}_S e^{-i\hat{H}t/\hbar}, \hat{H} \right] |\varphi_S(0)\rangle$$

In Heisenberg's picture, we define

$\hat{U}^\dagger(t) = \hat{U}^{-1}(t)$

$$|\varphi_H(t)\rangle = e^{i\hat{H}t/\hbar} |\varphi_S(t)\rangle \equiv |\varphi_S(0)\rangle$$

$$|\varphi_H\rangle \equiv |\varphi_S(0)\rangle$$

$$= e^{i\hat{H}t/\hbar} |\varphi_S(t)\rangle$$

$$\hat{F}_H = e^{i\hat{H}t/\hbar}\,\hat{F}_S\,e^{-i\hat{H}t/\hbar}$$

$$\hat{H}_H = e^{i\hat{H}t/\hbar}\,\underbrace{\hat{H}_S}_{\hat{H}}\,e^{-i\hat{H}t/\hbar}$$

$$= \hat{H}_S$$

$$\frac{d}{dt}\langle \varphi_H | \hat{F}_H | \psi_H \rangle \qquad \forall\ |\varphi_H\rangle, |\psi_H\rangle$$

$$= \langle \varphi_H | \frac{d\hat{F}_H}{dt} | \psi_H \rangle$$

$$= \langle \varphi_H | \left(\frac{\partial \hat{F}}{\partial t}\right)_H | \psi_H \rangle$$

$$+ \frac{1}{i\hbar}\langle \varphi_H | [\hat{F}_H, \hat{H}] | \psi_H \rangle$$

$$\boxed{\frac{d\hat{F}_H}{dt} = \left(\frac{\partial \hat{F}}{\partial t}\right)_H + \frac{1}{i\hbar}[\hat{F}_H, \hat{H}]}$$

Heisenberg's equation of motion

PHYSICAL RESULTS DO NOT DEPEND ON A PARTICULAR PICTURE

$\langle \varphi | \hat{F} | \chi \rangle$ does not depend on a picture $e^{-i\hat{H}t/\hbar}$

Proof:

$$\langle \varphi_H | \hat{F}_H | \chi_H \rangle = \langle \varphi_S | \underbrace{\left(e^{i\hat{H}t/\hbar}\right)^\dagger}_{}$$

$$\times \; \underbrace{e^{i\hat{H}t/\hbar}}\hat{F}_S\underbrace{e^{-i\hat{H}t/\hbar}\,e^{i\hat{H}t/\hbar}}_{=\,1}|\psi_S\rangle$$

$$= \langle \varphi_S|\hat{F}_S|\psi_S\rangle$$

In quantum mechanics,

$$\frac{d\hat{F}_H}{dt} = \left(\frac{\partial \hat{F}}{\partial t}\right)_H + \frac{1}{i\hbar}[\hat{F}_H, \hat{H}]$$

In classical mechanics,

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}$$

QUANTIZATION OF CLASSICAL MECHANICS

1. Write a classical equation of motion,

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}$$

2. Quantize it by replacing

F by $\hat{F}_H$

and

$\{ , \}$ by $\frac{1}{i\hbar}[,]$. $\checkmark$

We obtain (H)

$$\frac{d\hat{F}_H}{dt} = \left(\frac{\partial \hat{F}}{\partial t}\right)_H + \frac{1}{i\hbar}[\hat{F}_H, \hat{H}]$$

3. Solve Eq. (H) and calculate $\langle \varphi_H | \hat{F}_H | \psi_H \rangle$.

In this procedure,

$$\{q_k, q_l\} = 0, \quad \{p_k, p_l\} = 0,$$
$$\{q_k, p_l\} = \delta_{kl}$$

are replaced by

$$[\hat{q}_k^{(H)}, \hat{q}_l^{(H)}] = 0, \quad [\hat{p}_k^{(H)}, \hat{p}_l^{(H)}] = 0,$$
$$[\hat{q}_k^{(H)}, \hat{p}_l^{(H)}] = i\hbar\,\delta_{kl}. \quad \checkmark$$

These three relations remain the same in the Schrödinger picture.

$$i\hbar\delta_{kl} = [\hat{q}_k^{(H)}, \hat{p}_l^{(H)}] = \hat{q}_k^{(H)}\hat{p}_l^{(H)}$$

$$- \hat{p}_l^{(H)}\hat{q}_k^{(H)} = e^{i\hat{H}t/\hbar}\hat{q}_k^{(S)}e^{-i\hat{H}t/\hbar}$$

$$\times e^{i\hat{H}t/\hbar}\hat{p}_l^{(S)}e^{-i\hat{H}t/\hbar}$$

$$- e^{i\hat{H}t/\hbar}\hat{p}_l^{(S)}e^{-i\hat{H}t/\hbar}$$

$$\times e^{i\hat{H}t/\hbar}\hat{q}_k^{(S)}e^{-i\hat{H}t/\hbar}$$

$$= e^{i\hat{H}t/\hbar}\left(\hat{q}_k^{(S)}\hat{p}_l^{(S)} - \hat{p}_l^{(S)}\hat{q}_k^{(S)}\right)e^{-i\hat{H}t/\hbar}$$

$$i\hbar\delta_{kl} = [\hat{q}_k^{(H)}, \hat{p}_l^{(H)}]$$

$$= e^{i\hat{H}t/\hbar}[\hat{q}_k^{(S)}, \hat{p}_l^{(S)}]e^{-i\hat{H}t/\hbar}$$

$$[\hat{q}_k^{(S)}, \hat{p}_l^{(S)}] = e^{-i\hat{H}t/\hbar} (i\hbar\delta_{kl}) \times e^{i\hat{H}t/\hbar} = i\hbar\delta_{kl} \times \underbrace{e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar}}_{1}$$

$$[\hat{q}_k^{(S)}, \hat{p}_l^{(S)}] = i\hbar\delta_{kl} \, .$$

Example of the above quantization procedure

$$H = \frac{p^2}{2m} + V(x)$$

$$F = p \, , \quad \{x, p\} = 1$$

1. $$\frac{dp}{dt} = \cancel{\frac{\partial p}{\partial t}}^{\,0} + \{p, H\}$$

$$= \{p, H\} = \cancel{\left\{p, \frac{p^2}{2m}\right\}}^{\,0}$$

$$+ \{p, V\}$$

$$\frac{dp}{dt} = \{p, V\}$$

$$= \cancel{\frac{\partial p}{\partial x}}^{\,0}\cancel{\frac{\partial V}{\partial p}}^{\,0} - \cancel{\frac{\partial p}{\partial p}}^{\,1}\frac{\partial V}{\partial x}$$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} \leftarrow (F_x)$$

$$\boxed{\text{Here } \frac{\partial V}{\partial x} = \frac{dV}{dx}}$$

2. $$\frac{dp_H}{dt} = \cancel{\left(\frac{\partial p}{\partial t}\right)_H}^{\,0} + \frac{1}{i\hbar}[\hat{p}_H, \hat{H}]$$

$$[\hat{x}_H, \hat{p}_H] = i\hbar$$

$$[\hat{p}_H, \hat{H}] = \cancelto{0}{\left[\hat{p}_H, \frac{\hat{p}_H^2}{2m}\right]} + [\hat{p}_H, \hat{V}_H]$$

$$[\hat{p}_H, \hat{V}_H] = ?$$

We could calculate $[\hat{p}, \hat{V}]$ as follows:

$$[\hat{p}, \hat{V}] f(x) = \frac{\hbar}{i}\frac{d}{dx}(Vf)$$

$$- \frac{\hbar}{i} V \frac{df}{dx} = \frac{\hbar}{i}\frac{dV}{dx} f$$

$$+ \cancel{\frac{\hbar}{i} V \frac{df}{dx}} - \cancel{\frac{\hbar}{i} V \frac{df}{dx}}$$

$$= -i\hbar \frac{dV}{dx} f(x)$$

$$[\hat{p}_S, \hat{V}_S] = -i\hbar \frac{d\hat{V}_S}{dx}$$

$$[\hat{P}_H, \hat{V}_H] = -i\hbar \frac{d\hat{V}_H}{dx} \checkmark$$

But we will do it purely algebraically,

$$V = V_0 + \underbrace{\left.\frac{dV}{dx}\right|_{x=0}}_{V_1} x + \frac{1}{2!} \underbrace{\left.\frac{d^2V}{dx^2}\right|_{x=0}}_{V_2} x^2 + \cdots$$

$$V = \sum_{n=0}^{\infty} \frac{1}{n!} V_n x^n$$

$$V_n = \left.\frac{d^nV}{dx^n}\right|_{x=0}$$

$$[V, p] = ?$$

$$[V,p] = \left[\sum_{n=0}^{\infty} \frac{1}{n!} V_n x^n, p\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} V_n [x^n, p] \quad ✓$$

$$[x,p] = i\hbar \quad ✓$$

$$[x^2, p] = x[x,p] + [x,p]x$$

$$= 2i\hbar x$$

In general,

$$[x^n, p] = n i\hbar x^{n-1}. \quad ✓$$

straightforward proof by induction

$$[V,p] = \sum_{n=0}^{\infty} \frac{1}{n!} V_n \left(n i\hbar x^{n-1}\right)$$

$$= i\hbar \sum_{n=0}^{\infty} \frac{1}{n!} V_n \underbrace{\frac{dx^n}{dx}}_{n x^{n-1}}$$

$$= i\hbar \frac{d}{dx} \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} V_n x^n}_{V}$$

$$= i\hbar \frac{dV}{dx}$$

$$[V,p] = i\hbar \frac{dV}{dx}$$

$$[\hat{p}_H, \hat{V}_H] = -i\hbar \frac{d\hat{V}_H}{dx}$$

$$\frac{d\hat{p}_H}{dt} = \frac{1}{i\hbar}\left[\hat{p}_H, \hat{V}_H\right]$$

$$= \frac{1}{\cancel{i\hbar}}\left(-\cancel{i\hbar}\right)\frac{d\hat{V}_H}{dx}$$

$$\frac{d\hat{p}_H}{dt} = -\frac{d\hat{V}_H}{dx} \qquad \checkmark$$

3. $$\left\langle \varphi_H \middle| \frac{d\hat{p}_H}{dt} \middle| \psi_H \right\rangle$$

$$= -\left\langle \varphi_H \middle| \frac{d\hat{V}_H}{dx} \middle| \psi_H \right\rangle$$

$$\frac{d}{dt}\left\langle \varphi_H \middle| \hat{p}_H \middle| \psi_H \right\rangle$$

$$= -\left\langle \varphi_H \middle| \frac{d\hat{V}_H}{dx} \middle| \psi_H \right\rangle$$

Matrix elements of operators DO NOT DEPEND ON A PICTURE.

$$\frac{d}{dt}\langle \varphi_s | \hat{p}_s | \psi_s \rangle$$
$$= -\langle \varphi_s | \frac{d\hat{V}_s}{dx} | \psi_s \rangle$$

Ehrenfest's theorems

If $\varphi_s = \psi_s$, we obtain

$$\langle \varphi_s | \hat{p}_s | \varphi_s \rangle = \langle \hat{p}_s \rangle,$$
$$\langle \varphi_s | \frac{d\hat{V}_s}{dx} | \varphi_s \rangle = \left\langle \frac{d\hat{V}_s}{dx} \right\rangle$$
$$\frac{d}{dt}\langle \hat{p}_s \rangle = -\left\langle \frac{d\hat{V}_s}{dx} \right\rangle$$