HILBERT SPACE IS

- a linear space, $\alpha f + \beta g \in \mathcal{H}$ if $f, g \in \mathcal{H}$
- unitary, $(f, g) \in \mathbb{C}$
- metric, $\|f\| = \sqrt{(f,f)}$,

  $D[f,g] = \|f - g\|$
- complete with respect to $D[f,g]$

complete unitary space

HILBERT SPACE

IS ALSO A

BANACH SPACE

complete norm space

Example of $\mathcal{H}$:

$L^2(\Gamma)$,

where $\Gamma = \{(q_1, \ldots, q_f)\}$

$L^2(\Gamma)$ is a space functions

$\psi(q_1, \ldots, q_f)$ such

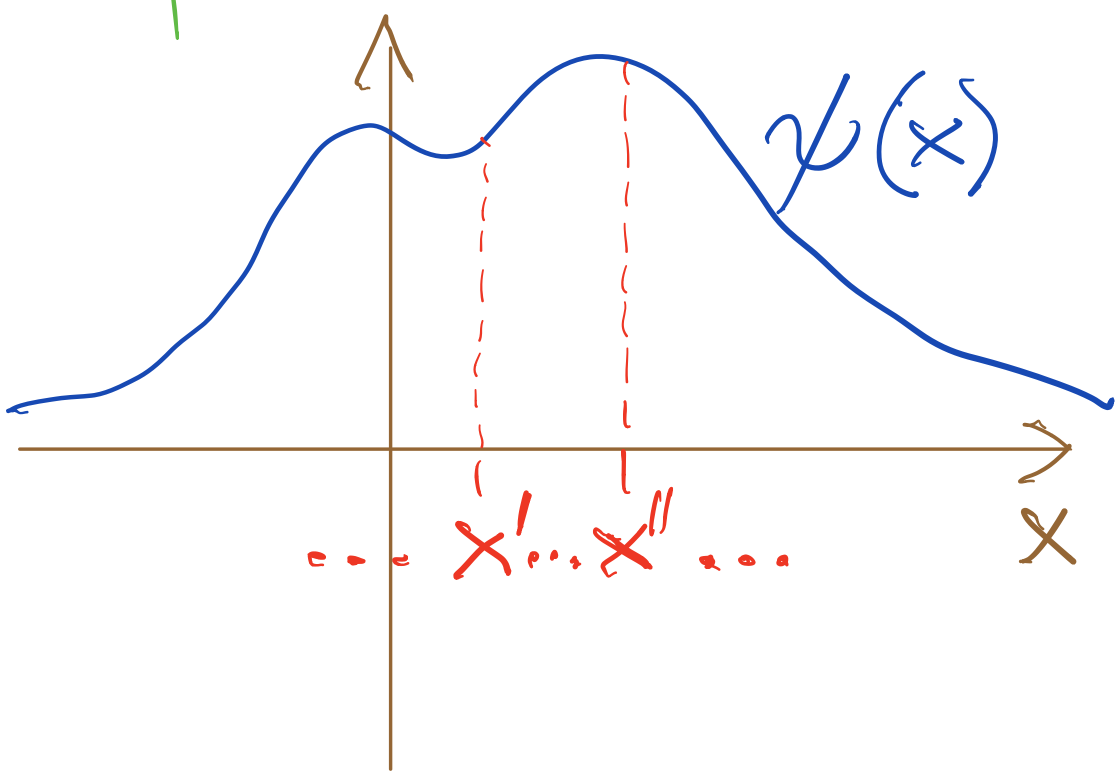
that

$$\int_\Gamma \psi^* \psi \, d\tau < \infty$$

$$\left(\int_{\Gamma} |\psi|^2 \, d\tau < \infty\right)$$

$$(\varphi, \psi) = \int_{\Gamma} \varphi^* \psi \, d\tau$$

(scalar product)



$$\psi = \begin{bmatrix} \vdots \\ \psi(x) \\ \vdots \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{x-th component of } \psi \\ \\ \end{array}$$

$$\varphi = \begin{bmatrix} \vdots \\ \varphi(x) \\ \vdots \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{x-th component of } \varphi \\ \\ \end{array}$$

$$\sum_x \varphi(x)^* \psi(x)$$ (as in $\vec{a}\cdot\vec{b} = \sum_i a_i^* b_i$)

$$\to \int_\Gamma \varphi^* \psi \, d\tau$$

Proof that (φ, ψ) is a scalar product:

(a) $(\psi, \varphi) = \int_\Gamma \psi^* \varphi \, d\tau$

$$= \int_\Gamma (\psi \varphi^*)^* \, d\tau$$

$$= \left(\int_\Gamma \psi \varphi^* \, d\tau \right)^*$$

$$= \left(\int_\Gamma \varphi^* \psi \, d\tau \right)^*$$

$$= \underline{(\varphi, \psi)^*}$$

(b) $\underline{(\varphi, \alpha_1 \psi_1 + \alpha_2 \psi_2)}$

$$= \int_\Gamma \varphi^* (\alpha_1 \psi_1 + \alpha_2 \psi_2) \, d\tau$$

$$= \alpha_1 \int_\Gamma \varphi^* \psi_1 \, d\tau$$

$$+ \alpha_2 \int_\Gamma \varphi^* \psi_2 \, d\tau$$

$$= \alpha_1 (\varphi, \psi_1) + \alpha_2 (\varphi, \psi_2)$$

(c) $$(\varphi, \varphi) = \int_\Gamma \varphi^* \varphi \, d\tau$$

$$= \int_\Gamma |\varphi|^2 \, d\tau \geq 0$$

$$(\varphi, \varphi) = 0$$

$$\Rightarrow \int_\Gamma |\varphi|^2 d\tau = 0$$

$$\Rightarrow \varphi \text{ is } 0$$

almost everywhere

(or everywhere)

such φ is a zero vector Θ in $L^2(\Gamma)$

Linearity of $L^2(\Gamma)$:

$$\int_\Gamma |\alpha\varphi + \beta\psi|^2 d\tau$$

$$= (\alpha\varphi + \beta\psi, \alpha\varphi + \beta\psi)$$

$$= \|\underbrace{\alpha\varphi}_{f} + \underbrace{\beta\psi}_{g}\|^2$$

$$= \|f + g\|^2$$

triangle inequality

$$\le (\|f\| + \|g\|)^2$$

$$= \|f\|^2 + \|g\|^2 + 2\|f\|\|g\|$$

$$= |\alpha|^2 \underbrace{\|\varphi\|^2}_{<\infty} + |\beta|^2 \underbrace{\|\psi\|^2}_{<\infty}$$

$$+ 2\,|\alpha| \underbrace{\|\varphi\|}_{<\infty}\, |\beta| \underbrace{\|\psi\|}_{<\infty}$$

$$\varphi, \psi \in L^2(\Gamma)$$

$$\|\varphi\|^2 = \int_\Gamma |\varphi|^2 \, d\tau < \infty \quad \checkmark$$

(finite)

$$\|\psi\|^2 = \int_\Gamma |\psi|^2 \, d\tau < \infty \quad \checkmark$$

(finite)

$$\int |\alpha\varphi + \beta\psi|^2 \, d\tau < \infty$$

$$\alpha\varphi + \beta\psi \in L^2(\Gamma)$$

$L^2(\Gamma)$ is UNITARY

$$\varphi_n \xrightarrow[n\to\infty]{} \varphi$$

$$\|\varphi_n\| = \sqrt{\int_\Gamma |\varphi_n|^2 d\tau} < \infty$$

$L^2(\Gamma)$ is also COMPLETE

(fundamental sequences in $L^2(\Gamma)$ converge in $L^2(\Gamma)$)

THIS MEANS THAT $L^2(\Gamma)$ IS A HILBERT SPACE

$$\|\varphi\| = (\varphi,\varphi)^{\frac{1}{2}}$$

$$= \sqrt{\int_\Gamma |\varphi|^2 d\tau}$$

norm of φ

$$\|\varphi\| = 1 \quad \checkmark$$

$$\int_{\Gamma} |\varphi|^2 d\tau = 1$$

(φ is normalized to unity)

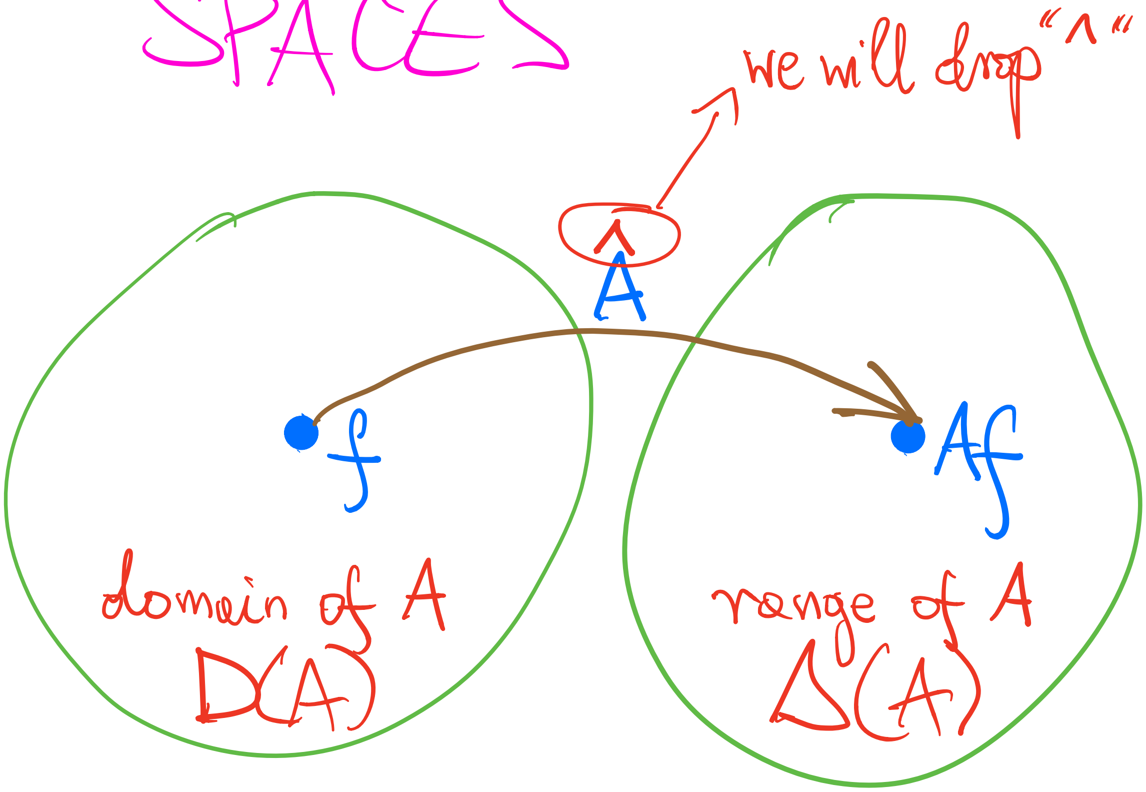
$$\int_{\Gamma} \psi^* \hat{F} \psi \, d\tau$$

($\|\psi\| = 1$)

$$= (\psi, \hat{F}\psi)$$

(expectation value of $\hat{F}$)

OPERATORS IN HILBERT SPACES



$A, B, \ldots$ – operators

$$\Delta(A) = \{ Af, f \in D(A) \}$$

An operator A is linear if its domain is a linear subspace (or manifold) of $\mathcal{H}$ and

$$A(\alpha f+\beta g) = \alpha Af + \beta Ag$$

$$\forall f,g \in \underbrace{D(A)}_{\text{linear subspace of } \mathcal{H}}$$

$$\alpha,\beta \in \mathbb{C}$$

- $(\alpha A+\beta B)f$
  $= \alpha(Af)+\beta(Bf)$
- $(AB)f = A(Bf)$
- $[A,B] = AB - BA$
  $[A,B]f = A(Bf)$
  $- B(Af)$

Bounded operators

A linear operator A is bounded if

$$\exists\ M < \infty \quad \forall\ f \in D(A)$$

$$\|Af\| \leq M \|f\|$$

A linear operator is continuous at f_0 if the condition $f_n \xrightarrow{n\to\infty} f_0$ yields

$$\lim_{n\to\infty} Af_n = Af_0$$

A is continuous if it is continuous at all $f \in D(A)$.

Every bounded linear operator is continuous and vice versa.

Assume that A is bounded,

$$f_n \xrightarrow[n\to\infty]{} f$$

$$\|Af_n - Af\|$$

$$= \|A(f_n - f)\|$$

$$\le \underset{<\infty}{M}\ \underbrace{\|f_n - f\|}_{\downarrow\, 0} \xrightarrow[n\to\infty]{} 0$$

$$Af_n \xrightarrow[n\to\infty]{} Af \quad \therefore$$

ADJOINT OPERATORS

$D(A)$ is dense in $\mathcal{H}$

A linear, $\quad \overline{D(A)} = \mathcal{H}$

It can be shown that there exists a companion operator B (with the domain $D(B)$) such that

$$(f, Ag) = (Bf, g)$$

$\forall$

$f \in D(B)$

$g \in D(A)$

We call B the adjoint operator to A and write

$$B = A^{+}$$

$A \leftrightarrow A^{+}$

$$(f, Ag) = (A^{+}f, g)$$

$f \in D(A^{+})$

$g \in D(A)$

turn-over rule (TOR)

If A is bounded and $D(A) = \mathcal{H}$, $A^\dagger$ is also bounded and $D(A^\dagger) = \mathcal{H}$.

Properties of "$\dagger$":

- $A^\dagger$ is linear

We will use (cf. the last lecture)

If $(f_1, g) = (f_2, g)$ $\forall g \in G$,

where G is dense in $\mathcal{H}$,

$(\overline{G} = \mathcal{H})$,

then $f_1 = f_2$.

$$\left(A^{+}(\alpha_1 f_1 + \alpha_2 f_2), g\right) \overset{\text{TOR}}{=} \left(\alpha_1 f_1 + \alpha_2 f_2, Ag\right)$$

$$\overset{(b)}{=} \alpha_1^*(f_1, Ag) + \alpha_2^*(f_2, Ag)$$

$$\overset{TOR}{=} \alpha_1^*(A^\dagger f_1, g) + \alpha_2^*(A^\dagger f_2, g)$$

$$\overset{(b)}{=} \underline{(\alpha_1 A^\dagger f_1 + \alpha_2 A^\dagger f_2, g)}$$

$\forall g \in D(A)$

$\overline{D(A)} = \mathcal{H}$

($D(A)$ dense in $\mathcal{H}$)

$$A^\dagger(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 A^\dagger f_1 + \alpha_2 A^\dagger f_2 \quad \therefore$$

- $(\alpha A+\beta B)^\dagger = \alpha^* A^\dagger + \beta^* B^\dagger$

$$((\alpha A+\beta B)^\dagger f, g)$$

$$\overset{\text{TOR}}{=} (f, (\alpha A+\beta B)g)$$

$$= (f, \alpha Ag + \beta Bg)$$

$$\overset{(b)}{=} \alpha(f, Ag) + \beta(f, Bg)$$

$$\overset{\text{TOR}}{=} \alpha(A^\dagger f, g) + \beta(B^\dagger f, g)$$

$$\overset{(b)}{=} (\alpha^* A^\dagger f + \beta^* B^\dagger f, g)$$

$$= ((\alpha^* A^\dagger + \beta^* B^\dagger) f, g)$$

True $\forall g$, so that

$$(\alpha A + \beta B)^\dagger f = (\alpha^* A^\dagger + \beta^* B^\dagger) f \qquad \forall f$$

$$(\alpha A+\beta B)^\dagger = \alpha^* A^\dagger + \beta^* B^\dagger \quad \therefore$$

- $$(AB)^\dagger = B^\dagger A^\dagger$$

$$\underline{((AB)^\dagger f, g)}$$

$$\overset{TOR}{=} (f, ABg) \quad \forall$$

$$\overset{TOR}{=} (A^\dagger f, Bg) \quad g$$

$$= (B^\dagger A^\dagger f, g)$$

$$(AB)^\dagger f = B^\dagger A^\dagger f \quad \forall f$$

$$(AB)^\dagger = B^\dagger A^\dagger \quad \checkmark$$

$$\therefore$$

- $(A^n)^\dagger = (A^\dagger)^n$

$n = 0, 1, 2, \ldots$

$$A^0 \equiv \mathbb{1}$$ the unit operator

$$\mathbb{1} f = f \quad \forall f$$

$$(A^2)^\dagger = (\overset{1}{A}\,\overset{2}{A})^\dagger$$

$$= \overset{2}{A}{}^\dagger \, \overset{1}{A}{}^\dagger$$

$$= (A^\dagger)^2 \quad , \quad n=2$$

• $(A^{-1})^t = (A^t)^{-1}$

$\Uparrow$

$$(f,g) = (f, \underbrace{A^{-1}A}_{\mathbb{1}} g)$$

$$\underset{\text{TOR}}{=} ((A^{-1})^t f, A g)$$

$$\underset{\text{TOR}}{=} (A^t (A^{-1})^t f, g)$$

$$\mathbb{1} f = A^t (A^{-1})^t f$$

$$A^\dagger (A^{-1})^\dagger = \mathbb{1}$$

$$(A^{-1})^\dagger A^\dagger = \mathbb{1}$$

$$(A^\dagger)^{-1} = (A^{-1})^\dagger \checkmark$$

$$\therefore$$

$$A^{-n} = (A^{-1})^n$$

$$n \geq 0$$

$$(A^n)^\dagger = (A^\dagger)^n$$

$$n = 0, \pm 1, \pm 2, \ldots$$

$$\boxed{\text{SYMMETRIC (HERMITIAN) AND SELF-ADJOINT OPERATORS}}$$

A linear operator A is said to be symmetric (Hermitian) if

(a) $\overline{D(A)} = \mathcal{H}$

($D(A)$ is dense in $\mathcal{H}$)

(b) ✓ $(Af, g) = (f, Ag)$ $\forall f, g \in D(A)$

A linear operator A is self-adjoint if $A = A^\dagger$.

$$(A^\dagger f, g) = (f, Ag)$$

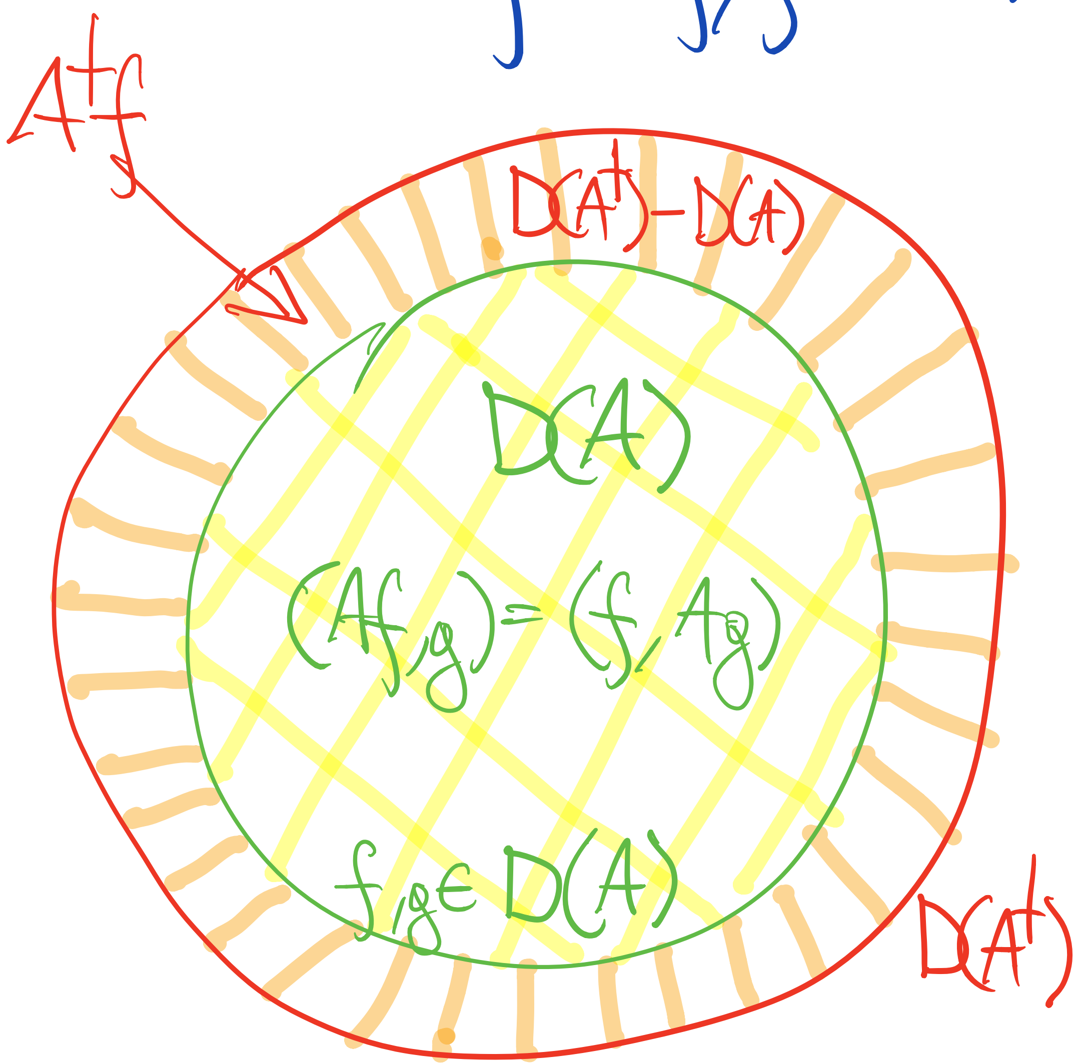
$$g \in D(A)$$

$$f \in D(A^\dagger)$$

For Hermitian A,

$$(\underset{\sim}{A}f, g) = (f, \underset{\sim}{A}g)$$

for $f, g \in D(A)$



$(Af, g) = (f, Ag)$ for $f, g \in D(A)$, BUT

$(A^\dagger f, g) = (f, Ag)$ for $f \in D(A^\dagger) - D(A)$, $g \in D(A)$

For Hermitian operators

$$(Af, g) = (f, Ag)$$

$\forall f, g \in D(A)$, but

$$(A^\dagger f, g) = (f, Ag)$$

for $f \in D(A^\dagger) - D(A)$, $g \in D(A)$, i.e.,

$A \neq A^\dagger$ if $D(A) \subset D(A^\dagger)$.

For Hermitian operators

$$Af = A^{\dagger}f, \quad f \in D(A)$$

$$D(A) \subseteq D(A^{\dagger})$$

(we write sometimes

$$A \subseteq A^{\dagger}).$$

A is self-adjoint if

$$Af = A^{\dagger}f, \quad f \in D(A)$$

and $D(A) = D(A^{\dagger})$.

TO PROVE THAT A IS SELF-ADJOINT WE MUST PROVE:

- A is Hermitian
- $D(A) = D(A^\dagger)$

All operators in QM must be

SELF-ADJOINT.

Every operator has a spectrum (set of complex number associated with an operator)

To calculate the spectrum, one must consider $(\lambda \mathbb{1} - A)^{-1}, \lambda \in \mathbb{C}$.

If $(\lambda \mathbb{1} - A)^{-1}$ exists and is bounded in $\mathcal{H}$, λ is called a regular value of A. All other values of λ comprise the spectrum of A.

In general, spectrum consists of

- point spectrum
- continuous spectrum
- residual spectrum

$\boxed{\text{point spectrum consists of discrete set of eigenvalues (if real, can be interpreted physically)}}$

$\lambda \in$ point spectrum

$Af = \lambda f, \ f \neq 0$

eigenvalue

$\boxed{\text{continuous spectrum is analogous to continuous spectra of operators representing observables (as in scattering states)}}$

(if real can be interpreted physically as well)

$$\boxed{\text{residual spectrum cannot have physical interpretation}}$$

Self-adjoint operators have only point and continuous spectra (residual spectra

are empty) and
both types of spectra
are subsets of real
numbers

Hermitian operators
have real eigenvalues
but their residual
spectra are not
empty unless they
are self-adjoint.

To make an operator $\hat{F}$ acceptable in QM we must prove that

- $\hat{F}$ is Hermitian
- $D(\hat{F}) = D(\hat{F}^{\dagger})$

(Kato)