CEM 991 **Fall 2023**

Assignment No. 8 (25 pts.) **Due Date: December 8, 2023 (Friday)**

1. (10 pts.) Use the generating function for the Hermite polynomials to evaluate
   (a) [5 pts.] $\langle u_n|\hat{x}^3|u_n\rangle$.
   (b) [5 pts.] $\langle u_n|\hat{x}^4|u_n\rangle$.

   Auxiliary formula: $\int_{-\infty}^{+\infty} x^{2n} e^{-ax^2} dx = \frac{(2n)!}{2^{2n} n!} \sqrt{\frac{\pi}{a^{2n+1}}}$, $a > 0$, $n = 0, 1, 2, \ldots$.

2. (7 pts.) Let $|n\rangle$ be a short-hand, Dirac-style, notation for the n-th eigenstate u_n of the one-dimensional quantum-mechanical harmonic oscillator ($n = 0, 1, \ldots$). Let ω be the angular frequency of the corresponding classical harmonic oscillator and m be its mass. Use the language of second quantization or occupation number representation, i.e., the formulas

$$\hat{x} = \left(\frac{\hbar}{2\omega m}\right)^{\frac{1}{2}} (a + a^\dagger),$$

$$\hat{p} = -i \left(\frac{\hbar \omega m}{2}\right)^{\frac{1}{2}} (a - a^\dagger),$$

$$a|n\rangle = n^{\frac{1}{2}} |n-1\rangle,$$

$$a^\dagger|n\rangle = (n+1)^{\frac{1}{2}} |n+1\rangle,$$

$$N|n\rangle = n|n\rangle,$$

   where $\hat{x}$, $\hat{p}$, a, and $a^\dagger$ are the coordinate, momentum, annihilation, and creation operators, respectively, and $N = a^\dagger a$ is the number operator, to calculate the expectation values of the kinetic and potential energy operators, $\hat{T} = \frac{\hat{p}^2}{2m}$ and $\hat{V} = \frac{1}{2} K \hat{x}^2$, respectively, where $K = m\omega^2$ is the force constant, in the n-th eigenstate $|n\rangle$. Compare the results with the total energy $E_n = \hbar\omega(n + \frac{1}{2})$.

3. (8 pts.) Let us designate the eigenstates of the one-dimensional harmonic oscillator, u_n, as $|n\rangle$, $n = 0, 1, \ldots$. The only non-zero matrix elements of the coordinate and momentum operators, $\hat{x}$ and $\hat{p}$, respectively, in an energy representation defined by the eigenstates $|n\rangle$ are

$$\langle n|\hat{x}|n+1\rangle = (n+1)^{1/2} \left(\frac{\hbar}{2m\omega}\right)^{1/2}, \tag{1}$$

$$\langle n|\hat{x}|n-1\rangle = n^{1/2} \left(\frac{\hbar}{2m\omega}\right)^{1/2}, \tag{2}$$

$$\langle n|\hat{p}|n+1\rangle = -i(n+1)^{1/2} \left(\frac{\hbar m\omega}{2}\right)^{1/2}, \tag{3}$$

$$\langle n|\hat{p}|n-1\rangle = in^{1/2}\left(\frac{\hbar m\omega}{2}\right)^{1/2}, \tag{4}$$

where m and ω designate the mass and the angular frequency of the classical harmonic oscillator. Use Eqs. (1)–(4) and the resolution of identity involving the eigenstates $|n\rangle$ to calculate matrix elements $\langle 0|\hat{H}|0\rangle$ and $\langle 1|\hat{H}|1\rangle$, where $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$ is the Hamiltonian of the harmonic oscillator. What should the values of $\langle 0|\hat{H}|0\rangle$ and $\langle 1|\hat{H}|1\rangle$ be? Comment on your findings.