CEM 991 Fall 2023

Assignment No. 7 (25 pts.) Due Date: November 29, 2023 (Wednesday)

1. (13 pts.) Let F and G be two dynamical variables and let $\hat{F}$ and $\hat{G}$ represent the corresponding quantum-mechanical operators. Let $|\phi\rangle$ be the normalized quantum state and let $\langle\hat{F}\rangle$ and $\langle\hat{G}\rangle$ represent the expectation values of F and G in the state described by $|\phi\rangle$. Let us define the following operators:

$$\Delta\hat{F} = \hat{F} - \langle\hat{F}\rangle, \tag{1}$$

$$\Delta\hat{G} = \hat{G} - \langle\hat{G}\rangle, \tag{2}$$

$$\hat{A} = \Delta\hat{F} + \imath\,\alpha\,\Delta\hat{G}, \tag{3}$$

where α is a real variable and $\imath$ is an imaginary unit. Let us also introduce the auxiliary function

$$I(\alpha) = \langle\hat{A}^\dagger\hat{A}\rangle = ||\hat{A}|\phi\rangle||^2. \tag{4}$$

As discussed in class, the condition

$$I(\alpha) \geq 0, \quad \text{for all } \alpha, \tag{5}$$

leads to the Heisenberg relation,

$$(\Delta F)^2\,(\Delta G)^2 \geq -\frac{1}{4}\langle[\hat{F},\hat{G}]\rangle^2, \tag{6}$$

where ΔF and ΔG are the uncertainties of F and G in a state described by $|\phi\rangle$.

(a) [5 pts.] Let us assume that $|\phi\rangle$ is a *minimum wave packet*, so that

$$(\Delta F)^2\,(\Delta G)^2 = -\frac{1}{4}\langle[\hat{F},\hat{G}]\rangle^2 \tag{7}$$

in a quantum state described by $|\phi\rangle$. Show that Eq. (7) is equivalent to the condition

$$I(\alpha_0) = 0, \tag{8}$$

where

$$\alpha_0 = -\frac{\imath\langle[\hat{F},\hat{G}]\rangle}{2(\Delta G)^2}. \tag{9}$$

(b) [3 pts.] Show that condition (8), with α_0 defined by Eq. (9), is equivalent to an equation

$$\left[\Delta\hat{F} + \frac{\langle[\hat{F},\hat{G}]\rangle}{2(\Delta G)^2}\Delta\hat{G}\right]|\phi\rangle = 0. \tag{10}$$

Equation (10) can be viewed as a fundamental equation determining the *minimum wave packet* $|\phi\rangle$.

(c) [5 pts.] Let us consider the one-dimensional *minimum wave packet* $|\phi\rangle$ for $F = p_x$ and $G = x$, so that

$$\Delta x\, \Delta p_x = \frac{\hbar}{2}. \tag{11}$$

Show that Eq. (10) for $F = p_x$ and $G = x$ is equivalent to a differential equation

$$\left[\frac{d}{dx} + \frac{x - x_0}{2(\Delta x)^2} - \frac{\imath}{\hbar} p_0\right] \phi(x) = 0, \tag{12}$$

where $x_0 = \langle \hat{x} \rangle$ and $p_0 = \langle \hat{p}_x \rangle$. Find the solution of Eq. (12) (you are not required to normalize $\phi(x)$).

2. (8 pts.) One can show that for any eigenstate u_n of the linear harmonic oscillator,

$$\langle \hat{V} \rangle_n \equiv \langle u_n | \hat{V} | u_n \rangle = \tfrac{1}{2} E_n = \tfrac{1}{2} \hbar \omega_c (n + \tfrac{1}{2}). \tag{13}$$

Here, $\hat{V} = \frac{1}{2} K \hat{x}^2$ is the potential energy operator, $K = m\omega_c^2$ is the force constant, ω_c is the angular frequency of the classical harmonic oscillator, and E_n is the energy of the n-th state u_n. Use Eq. (13) and the fact that $\langle \hat{x} \rangle_n \equiv \langle u_n | \hat{x} | u_n \rangle$ and $\langle \hat{p_x} \rangle_n \equiv \langle u_n | \hat{p_x} | u_n \rangle$ vanish to show that the uncertainty product $\Delta x\, \Delta p_x$ equals $(n + \frac{1}{2})\hbar$ for the state described by u_n. What can you say about the $n = 0$ state from the point of view of Problem 1?

3. (4 pts.) Show that the following operator identity is true,

$$Q - \frac{d}{dQ} = -e^{\frac{1}{2}Q^2} \frac{d}{dQ}\, e^{-\frac{1}{2}Q^2}, \tag{14}$$

where Q is a dimensionless variable.