CEM 991 **Fall 2023**

Assignment No. 6 (25 pts.) **Due Date: November 17, 2023 (Friday)**

1. (4 pts.) Use the following representation of the Dirac delta function [cf. E. Merzbacher, *Quantum Mechanics*, 3rd edition (Wiley, New York, 1998), Appendix, pp. 630-634]:

$$\delta(y) = \lim_{n\to\infty} \frac{\sin ny}{\pi y},$$

to show that

$$\int_{-\infty}^{+\infty} e^{ikx} dx = 2\pi\delta(k).$$

Hint: $\int_{-\infty}^{+\infty} e^{ikx} dx = \lim_{a\to\infty} \int_{-a}^{+a} e^{ikx} dx.$

2. (8 pts.) Let $\delta(x)$ be the Dirac delta function. Show that

(a) $\delta(x) = \delta(-x)$.

(b) $x\delta(x) = 0$.

(c) $\delta(ax) = a^{-1}\delta(x)$, $a > 0$.

(d) $\int_{-\infty}^{\infty} \delta(a-x)\delta(x-b)\, dx = \delta(a-b)$.

Recall that the Dirac delta function is not an ordinary function. The meaning of each of the above equations is that the corresponding left- and right-hand sides, when used as multiplying factors under an integral sign, lead to the same result. For example, in order to prove (a), i.e., $\delta(x) = \delta(-x)$, you should prove that $\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(x)\delta(-x)dx$ for any function $f(x)$.

3. (8 pts.) The z component of the angular momentum operator $\hat{\mathbf{L}}$ is represented by the operator

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi},$$

where ϕ is a polar angle. The normalized eigenfunctions of $\hat{L}_z$ are

$$\phi_m = \frac{1}{\sqrt{2\pi}} e^{im\phi},$$

and the corresponding eigenvalues are $m\hbar$, where $m = 0, \pm 1, \pm 2, \ldots$. Consider a state described by the wave function

$$\psi(\phi) = C\cos^2\phi, \tag{1}$$

where C is a constant. Calculate the possible values $m\hbar$ of L_z and their probabilities $P(m\hbar)$ in a state described by the wave function $\psi(\phi)$, Eq. (1). Calculate the expectation value

$$\langle \hat{L}_z \rangle = \frac{\int_0^{2\pi} \psi^*(\phi) \hat{L}_z \psi(\phi) d\phi}{\int_0^{2\pi} \psi^*(\phi) \psi(\phi) d\phi}$$

and compare the result with the expectation value of $\hat{L}_z$ defined as

$$\langle \hat{L}_z \rangle = \sum_m m\hbar P(m\hbar).$$

Hint: Use the equations $\cos^2 \phi = \frac{1}{2}(1 + \cos 2\phi)$ and $\cos \alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$.

4. (5 pts.) Consider a one-particle system in a state described by a spherically-symmetric wave function $\psi = \psi(r)$ (r is the distance between particle and the origin). Show that ψ is a simultaneous eigenfunction of all three angular momentum operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$, in spite of the fact that $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ do not commute. Find the corresponding eigenvalues. *This example demonstrates that non-commuting operators may have common eigenstates.*

SOLUTIONS

1. $$\int_{-\infty}^{+\infty} e^{ikx}\,dx = \lim_{a\to\infty} \int_{-a}^{+a} e^{ikx}\,dx$$

$$= \lim_{a\to\infty} \frac{e^{ika} - e^{-ika}}{ik}$$

$$= 2\lim_{a\to\infty} \frac{e^{ika} - e^{-ika}}{2ik}.$$

From the Euler formula for $e^{\pm ika}$,

$$\frac{e^{ika} - e^{-ika}}{2i} = \sin ka.$$

Thus,

$$\int_{-\infty}^{\infty} e^{ikx}\,dx = 2\lim_{a\to\infty} \frac{\sin ka}{k}$$

$$= 2\pi \lim_{a\to\infty} \frac{\sin ka}{\pi k} = \underline{\underline{2\pi\,\delta(k)}}.$$

2. (a) We must prove that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \int_{-\infty}^{\infty} f(x)\delta(-x)\,dx.$$

The left-hand side (LHS) is

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0).$$

The right-hand side (RHS) is

$$\int_{-\infty}^{\infty} f(x)\,\delta(-x)\,dx \underset{dy=-dx}{\overset{y=-x}{=}}$$

$$= -\int_{+\infty}^{-\infty} f(-y)\,\delta(y)\,dy$$

$$= \int_{-\infty}^{\infty} f(-y)\,\delta(y)\,dy = f(0).$$

$$\therefore \quad LHS = RHS.$$

(b) We must prove that

$$\int_{-\infty}^{\infty} f(x)\, x\, \delta(x)\, dx = 0.$$

Let us define

$$g(x) = x\, f(x).$$

We have, $\displaystyle\int_{-\infty}^{\infty} g(x)\,\delta(x)\,dx = g(0),$

where $g(0) = 0 \cdot f(0) = 0.$

Thus, $\displaystyle\int_{-\infty}^{\infty} g(x)\,\delta(x)\,dx = \int_{-\infty}^{\infty} f(x)\,x\,\delta(x)\,dx$

$= 0$ for all functions $f(x)$.

$\therefore$

(c) We must prove that

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

LHS: $\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx \quad \underline{\underline{y=ax}} \quad dy = a\,dx,\ a>0$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)\,dy$$

$$= \frac{1}{a}\,f\left(\frac{0}{a}\right) = \frac{1}{a}\,f(0).$$

RHS:

$$\frac{1}{a}\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = \frac{1}{a}\,f(0).$$

$$\underline{LHS = RHS}$$

(d) We must prove that

$$\int_{-\infty}^{\infty} f(a) \left(\int_{-\infty}^{\infty} \delta(a-x)\,\delta(x-b)\, dx \right) da$$

$$= \int_{-\infty}^{\infty} f(a)\, \delta(a-b)\, da$$

LHS: $$\int_{-\infty}^{\infty} f(a) \left(\int_{-\infty}^{\infty} \delta(a-x)\,\delta(x-b)\, dx \right) da$$

$$= \int_{-\infty}^{\infty} \delta(x-b) \left(\int_{-\infty}^{\infty} f(a)\, \delta(a-x)\, da \right) dx$$

$$= \int_{-\infty}^{\infty} \delta(x-b)\, f(x)\, dx = f(b).$$

RHS: $$\int_{-\infty}^{\infty} f(a)\, \delta(a-b)\, da = f(b).$$

$\boxed{LHS = RHS}$

3. In general, if we have an operator $\hat{A}$ and know its eigenvalues a_m and eigenfunctions u_m, we can always write

$$\psi = \sum_m c_m u_m ,$$

for any state ψ (let us assume that ψ is normalized). The possible values of $\hat{A}$ in a state described by ψ are characterized by the probabilities $P(a_m) = |c_m|^2$. Clearly, $\sum_m P(a_m) = 1$.

In this problem, we have:

$$\hat{A} = \hat{L}_z ,$$

$$u_m = \frac{1}{\sqrt{2\pi}} e^{im\varphi} ,$$

$$a_m = m\hbar \qquad (m = 0, \pm 1, \pm 2, \dots),$$

and

$$\psi(\varphi) = C\cos^2\varphi \; .$$

Using the formula

$$\cos^2\varphi = \frac{1+\cos 2\varphi}{2} \; ,$$

we obtain

$$\psi(\varphi) = C\left(\frac{1}{2} + \frac{1}{2}\cos 2\varphi\right) =$$

$$= C\left(\frac{1}{2} + \frac{1}{4}e^{2i\varphi} + \frac{1}{4}e^{-2i\varphi}\right)$$

$$= \sqrt{\frac{\pi}{2}}\, C \underbrace{\frac{1}{\sqrt{2\pi}}}_{m=0} + \frac{1}{2}\sqrt{\frac{\pi}{2}}\, C \underbrace{\frac{1}{\sqrt{2\pi}} e^{2i\varphi}}_{m=2}$$

$$+ \frac{1}{2}\sqrt{\frac{\pi}{2}}\, C \underbrace{\frac{1}{\sqrt{2\pi}} e^{-2i\varphi}}_{m=-2}$$

$$= c_0 u_0 + c_2 u_2 + c_{-2} u_{-2} \, ,$$

where

$$c_0 = \sqrt{\frac{\pi}{2}}\, C \, , \quad c_2 = c_{-2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\, C \, . \tag{1}$$

Since

$$\sum_m |c_m|^2 = 1, \quad \text{we obtain}$$

$$|c_0|^2 + |c_2|^2 + |c_{-2}|^2 = 1, \quad \text{so that}$$

$$\frac{\pi}{2} C^2 \left(1 + \frac{1}{4} + \frac{1}{4}\right) = 1 . \quad \text{This means}$$

that

$$\frac{\pi}{2} C^2 \cdot \frac{3}{2} = 1 \quad \text{and}$$

$$C = \sqrt{\frac{4}{3\pi}} = \frac{2}{\sqrt{3\pi}} . \quad (2)$$

Substitution of (2) into (1) gives us

$$c_0 = \sqrt{\frac{2}{3}}, \quad c_2 = c_{-2} = \frac{1}{\sqrt{6}}, \quad \text{so}$$

that

$$P(m\hbar) = |c_m|^2 \quad \text{are,}$$

$$P(0) = |c_0|^2 = \frac{2}{3},$$

$$P(2\hbar) = P(-2\hbar) = |c_2|^2 = |c_{-2}|^2 = \frac{1}{6} .$$

The possible values of L_z are 0, $2\hbar$, and $-2\hbar$, and the corresponding probabilities are $\frac{2}{3}$, $\frac{1}{6}$, and $\frac{1}{6}$, respectively.

Clearly, $\sum_m P(m\hbar) = \frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 1$.

Let us compute

$$\langle \hat{L}_z \rangle = \sum_m m\hbar \, P(m\hbar)$$

(in general, $\langle \hat{A} \rangle = \sum_m a_m P(a_m)$).

We obtain,

$$\langle \hat{L}_z \rangle = \underbrace{0\hbar \frac{2}{3}}_{m=0} + \underbrace{2\hbar \frac{1}{6}}_{m=2} + \underbrace{(-2\hbar)\frac{1}{6}}_{m=-2} = 0 .$$

Let us try to calculate

$$\langle \hat{L}_z \rangle = \frac{\int_0^{2\pi} \psi^*(\varphi)\hat{L}_z \psi(\varphi)\, d\varphi}{\int_0^{2\pi} \psi^*(\varphi)\psi(\varphi)\, d\varphi} .$$

$$\int_0^{2\pi} \psi^*(\varphi)\psi(\varphi)\, d\varphi = |C|^2 \int_0^{2\pi} \cos^4\varphi \, d\varphi .$$

$$\int_0^{2\pi} \psi^*(\varphi)\hat{L}_z \psi(\varphi)\, d\varphi = |C|^2 \int_0^{2\pi} \cos^2\varphi \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\cos^2\varphi \, d\varphi$$

$$= -2|C|^2 \frac{\hbar}{i} \int_0^{2\pi} \cos^3\varphi \sin\varphi \, d\varphi$$

[redacted]

$$= 0 , \quad \text{since}$$

$$\int_0^{2\pi} \cos^3\varphi \sin\varphi \, d\varphi = -\int_0^{2\pi} \cos^3\varphi \, d(\cos\varphi) =$$

$$= -\left.\frac{\cos^4\varphi}{4}\right|_0^{2\pi} = 0 .$$

Thus,

$$\langle \hat{L}_z \rangle = 0,$$

independent of the method used to calculate $\langle \hat{L}_z \rangle$. $\therefore$

4. We know that

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right),$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Let us consider $\hat{L}_x \psi(r)$, where $\psi(r)$ is a spherically symmetric wave function.

We obtain,

$$\hat{L}_x \psi(r) = \frac{\hbar}{i}\left[y \frac{\partial \psi(r)}{\partial z} - z \frac{\partial \psi(r)}{\partial y} \right].$$

Since $r = (x^2+y^2+z^2)^{\frac{1}{2}}$, we can write

$$\frac{\partial \psi(r)}{\partial z} = \frac{d\psi(r)}{dr} \frac{\partial r}{\partial z} = \frac{d\psi(r)}{dr}$$

$$\times \frac{\partial}{\partial z}(x^2+y^2+z^2)^{\frac{1}{2}}$$

$$= \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}} \cancel{2} z \frac{d\psi(r)}{dr}$$

$$= \frac{z}{r} \frac{d\psi(r)}{dr}.$$

In other words,

$$\frac{\partial \psi(r)}{\partial z} = \frac{z}{r} \frac{d\psi(r)}{dr}.$$

Similarly, $$\frac{\partial \psi(r)}{\partial y} = \frac{y}{r} \frac{d\psi(r)}{dr}.$$

Thus,

$$\hat{L}_x \psi(r) = \frac{\hbar}{i}\left[y\frac{z}{r}\frac{d\psi(r)}{dr} - z\frac{y}{r}\frac{d\psi(r)}{dr}\right]$$

$$= \frac{\hbar}{i}\left(\frac{yz - yz}{r}\right)\frac{d\psi(r)}{dr} = 0.$$

We can repeat the analogous considerations for $\hat{L}_y \psi(r)$ and $\hat{L}_z \psi(r)$ and show that $\hat{L}_y \psi(r) = 0$ and $\hat{L}_z \psi(r) = 0$.

In other words,

$\hat{L}_x \psi(r) = 0 \cdot \psi(r)$, $\hat{L}_y \psi(r) = 0 \cdot \psi(r)$,

and $\hat{L}_z \psi(r) = 0 \cdot \psi(r)$, which means that $\psi(r)$ is an eigenfunction of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$. The corresponding eigenvalues are 0.

Interestingly, the operator for the finite rotation $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ $[\alpha_x, \alpha_y, \alpha_z$

are rotation angles corresponding to rotations about x, y, and z, respectively], which allows us to rotate any function $f(x,y,z)$, can be given the form,

$$\hat{U}_{\vec{\alpha}} = e^{-i\vec{\alpha}\cdot\hat{\vec{L}}/\hbar},$$

where $\hat{\vec{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$.

Since $\psi(r)$ is spherically symmetric,

$$\hat{U}_{\vec{\alpha}}\,\psi(r) = \psi(r)$$ for every $\vec{\alpha}$. This means that

$$e^{-i\vec{\alpha}\cdot\hat{\vec{L}}/\hbar}\,\psi(r) = \psi(r),$$

for all $\vec{\alpha}$'s, which implies that

$$\hat{L}_{x(y,z)}\,\psi(r) = 0.$$

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