

1. (7 pts.) Find the point spectrum (eigenvalues) of the projection operator P_G , where G is the proper subspace of the Hilbert space $\mathcal{H}$ (i.e. $G \subset \mathcal{H}$, $G \neq \mathcal{H}$, $G \neq \{\Theta\}$, where Θ is the zero vector). Find the corresponding eigenspaces (sets of eigenvectors corresponding to eigenvalues of P_G).
2. (9 pts.) The operator U is called *unitary* if (i) its domain is the entire Hilbert space ($D(U) = \mathcal{H}$) (ii) its range is the entire Hilbert space ($\Delta(U) = \mathcal{H}$), and (iii) for every $f, g \in \mathcal{H}$,

$$(Uf, Ug) = (f, g). \quad (1)$$

(a) Show that $U^{-1} = U^\dagger$.

(b) The above definition of a unitary operator does not mention that unitary operators are also linear operators, and yet it can be shown that they are linear. In particular,

$$U(\alpha f) = \alpha Uf, \quad (2)$$

where α is a scalar (real or complex number). Use the inner product expression $(U(\alpha f) - \alpha Uf, U(\alpha f) - \alpha Uf)$, the bilinearity and positive-definiteness of the inner product, and the property represented by Eq. (1) to show that Eq. (2) is indeed true.

(c) Show that eigenvectors f_1 and f_2 belonging to two different eigenvalues λ_1 and λ_2 of a unitary operator U are orthogonal.

Hint: Consider (Uf_1, Uf_2) , use the fact that each eigenvalue of a unitary operator U has an absolute value of 1 [see Problem 3 (b) in Assignment No. 4], and note that a complex number λ , whose absolute value is 1, can always be written as $\lambda = e^{i\alpha}$, where $0 \leq \alpha < 2\pi$.

3. (9 pts.) A linear operator A is called *skew-symmetric* if $A^\dagger = -A$.

(a) Prove that the point spectrum (a set of eigenvalues) of the skew-symmetric operator is a subset of the imaginary axis.

(b) Prove that the eigenvectors of a skew-symmetric operator A that correspond to two different eigenvalues of A are orthogonal.

(c) Prove that a differential operator

$$\frac{\partial}{\partial \phi}$$

is skew-symmetric (ϕ is a polar angle, $\phi \in [0, 2\pi]$).

1. METHOD 1:

Consider the eigenvalue problem,

$$P_G h = \lambda h, \quad h \neq 0.$$

In general,

$$h = g + f,$$

where $g \in G$, $f \in G^\perp$.

We obtain,

$$P_G h = P_G g + \underset{\substack{\uparrow \\ \textcircled{+}}}{P_G f} = g = \lambda h$$

$$= \lambda (g + f) = \lambda g + \lambda f,$$

so that

$$g = \lambda g + \lambda f$$

or

$$(1 - \lambda)g = \lambda f.$$

Now,

$$(1 - \lambda)g \in G$$

$$\lambda f \in G^\perp.$$

Since the only vector shared by G and $G^\perp$ is the zero vector $\{0\}$, we must have

$$\lambda f = \{0\} = (1-\lambda)g,$$

which means that either

- $\lambda = 0$, or

- $\lambda \neq 0$, $f = \{0\}$.

$$\boxed{\lambda = 0}$$

In this case, $(1-\lambda)g = g = \{0\},$

which gives $h = g + f = f \in G^\perp,$

$$P_G h = 0h, \quad h \in G^\perp.$$

$\lambda = 0$ is an eigenvalue, $G^\perp$ is an eigenspace.

$$\boxed{\lambda \neq 0, f = \oplus}$$

In this case,

$$h = g + f = g \in G,$$

$$P_G h = P_G g = g = h = 1 \cdot h,$$

so that $\lambda = 1, h \in G$.

$\lambda = 1$ is an eigenvalue, G is the corresponding eigenspace.

The point spectrum consists of two eigenvalues, $\lambda = 0$ and $\lambda = 1$. The corresponding eigenspaces are $G^\perp$ and G , respectively.

METHOD 2:

$$P_G h = \lambda h, \quad h \neq \oplus.$$

We know that $P_G^2 = P_G$, so that

$$P_G^2 h = P_G(\lambda h) = \lambda P_G h = \lambda^2 h$$

and, at the same time,

$$P_G^2 h = P_G h = \lambda h.$$

We obtain,

$$\lambda^2 h = \lambda h, \quad h \neq \oplus,$$

so that

$$\lambda^2 = \lambda,$$

which means that $\lambda = 0$ or 1 .

The case $\lambda = 0$ gives

$$P_G h = \oplus, \text{ which means that } h \in G^\perp.$$

The case $\lambda = 1$ gives

$$P_G h = h, \text{ which means that } h \in G.$$

$\lambda = 0$ and $\lambda = 1$ are eigenvalues, $G^\perp$ and G are the corresponding eigenspaces.

2.

(a)

$$(\overleftarrow{Uf}, Ug) = (f, g)$$

$$\forall f, g \in \mathcal{H}$$

$\downarrow$ adjoint rule

$$(U^+ Uf, g) = (f, g)$$

$$\forall f, g \in \mathcal{H}$$

Thus,

$$U^+ U = I.$$

Since the domain and range of U is the entire Hilbert space, U^{-1} exists and

$$U^{-1} U = I.$$

Thus,

$$U^{-1} = U^+.$$

$$(b) \quad (U(\alpha f) - \alpha Uf, U(\alpha f) - \alpha Uf)$$

bilinearity
of inner product

$$= (\alpha f, \alpha f) - \alpha (Uf, U(\alpha f)) - \alpha^* (Uf, U(\alpha f)) + \alpha^* \alpha (Uf, Uf).$$

From Eq. (1), we obtain (after dropping U s),

$$\begin{aligned} & (\alpha f, \alpha f) - \alpha (\alpha f, f) - \alpha^* (f, \alpha f) \\ & + \alpha^* \alpha (f, f) = \cancel{\alpha^* \alpha (f, f)} - \cancel{\alpha \alpha^* (f, f)} \\ & - \cancel{\alpha^* \alpha (f, f)} + \cancel{\alpha^* \alpha (f, f)} = 0. \end{aligned}$$

Thus, $(U(\alpha f) - \alpha Uf, U(\alpha f) - \alpha Uf) = 0$.

In general, the condition $(g, g) = 0$ implies that $g = \mathbf{0}$ (zero vector). Thus,

$$U(\alpha f) - \alpha Uf = \mathbf{0} \quad \text{or}$$

$$\underline{U(\alpha f) = \alpha Uf}, \text{ which is Eq. (2).}$$

$$(c) \quad \begin{aligned} Uf_1 &= \lambda_1 f_1, \\ Uf_2 &= \lambda_2 f_2, \quad \lambda_2 \neq \lambda_1. \end{aligned}$$

$$\begin{aligned} (Uf_1, Uf_2) &= (\lambda_1 f_1, \lambda_2 f_2) \\ &= \lambda_1^* \lambda_2 (f_1, f_2). \end{aligned}$$

On the other hand,

$$(Uf_1, Uf_2) = (f_1, f_2).$$

We obtain,

$$\lambda_1^* \lambda_2 (f_1, f_2) = (f_1, f_2).$$

In Assignment 4, we proved that

(Problem 3(b)) $|\lambda_1| = |\lambda_2| = 1$, so that

$$\lambda_1 = e^{i\alpha_1}, \quad \lambda_2 = e^{i\alpha_2}, \quad \text{for}$$

some $0 \leq \alpha_1, \alpha_2 < 2\pi$.

Suppose $(f_1, f_2) \neq 0$. In this case,

$$\lambda_1^* \lambda_2 = e^{i(\alpha_2 - \alpha_1)} = 1.$$

Since both α_1 and α_2 belong to the interval $[0, 2\pi)$,

$e^{i(\alpha_2 - \alpha_1)} = 1$ implies that

$\alpha_2 - \alpha_1 = 0$ or that $\alpha_1 = \alpha_2$.

Thus, $\lambda_1 = \lambda_2$, contrary to our assumption that $\lambda_1 \neq \lambda_2$.

We cannot assume that $(f_1, f_2) \neq 0$.

In consequence,

$$(f_1, f_2) = 0 \quad \text{when } \lambda_1 \neq \lambda_2.$$

3. (a) Consider an eigenvalue problem

$$Af = \lambda f \quad (f - \text{eigenvector}, \lambda - \text{eigenvalue}).$$

$$(f, Af) = (f, \lambda f) = \lambda (f, f) = \lambda \|f\|^2$$

$$\begin{aligned} \parallel \\ (A^* f, f) &= (-Af, f) = -(\lambda f, f) = \\ &= -\lambda^* (f, f) = -\lambda^* \|f\|^2. \end{aligned}$$

Thus, $\lambda \|f\|^2 = -\lambda^* \|f\|^2$. Since

$f \neq 0$ and $\|f\| > 0$, we have

$$\lambda = -\lambda^*.$$

If $\lambda = a + bi$, where $a = \operatorname{Re} \lambda$, $b = \operatorname{Im} \lambda$,

we can write

$$a + bi = -(a - bi) = -a + b\bar{i}.$$

Thus,

$$a = -a \quad \text{or} \quad 2a = 0, \text{ i.e., } a = 0.$$

$\lambda = bi \rightarrow$ belongs to the imaginary axis.

(b) Consider

$$Af_1 = \lambda_1 f_1, \quad Af_2 = \lambda_2 f_2,$$

$$\lambda_1 \neq \lambda_2.$$

We obtain

$$(f_1, Af_2) = (f_1, \lambda_2 f_2) = \lambda_2 (f_1, f_2)$$

$$(A^t f_1, f_2) = (-Af_1, f_2) = -\lambda_1^* (f_1, f_2)$$

We already know from (a) that

λ_1 and λ_2 belong to the imaginary axis.

In particular,

$$\lambda_1 = -\lambda_1^*.$$

Thus,

$$\lambda_2(f_1, f_2) = -\lambda_1^*(f_1, f_2) = \lambda_1(f_1, f_2),$$

or

$$(\lambda_1 - \lambda_2)(f_1, f_2) = 0, \text{ so that}$$

either $\lambda_1 - \lambda_2 = 0$ or $(f_1, f_2) = 0$.

The former is not possible, since $\lambda_1 \neq \lambda_2$.

Thus,

$$(f_1, f_2) = 0.$$

(c) We must prove that

$$\left(\frac{\partial}{\partial \varphi}\right)^{\dagger} = -\frac{\partial}{\partial \varphi}.$$

In general, $\hat{F}^{\dagger}$ is defined by

$$\int_{\Gamma} f^* \hat{F} g d\tau = \int_{\Gamma} (\hat{F}^{\dagger} f)^* g d\tau.$$

In our case, $\hat{F} = \frac{\partial}{\partial \varphi}$, $\Gamma = [0, 2\pi]$.

$$\int_0^{2\pi} f^* \frac{\partial g}{\partial \varphi} d\varphi \stackrel{\substack{\text{integrating} \\ \text{by parts}}}{=} f^* g \Big|_{\varphi=0}^{\varphi=2\pi}$$

$$- \int_0^{2\pi} \left(\frac{\partial f}{\partial \varphi} \right)^* g d\varphi$$

$f(0) = f(2\pi)$, $g(0) = g(2\pi)$. Thus,

$$\int_0^{2\pi} f^* \frac{\partial g}{\partial \varphi} d\varphi = - \int_0^{2\pi} \left(\frac{\partial f}{\partial \varphi} \right)^* g d\varphi$$

$$= \int_0^{2\pi} \left(-\frac{\partial f}{\partial \varphi} \right)^* g d\varphi \equiv \int_0^{2\pi} \left[\left(\frac{\partial}{\partial \varphi} \right)^{\dagger} f \right]^* g d\varphi.$$

Thus, $\left(\frac{\partial}{\partial \varphi} \right)^{\dagger} = -\frac{\partial}{\partial \varphi}$.

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