CEM 991 **Fall 2023**

Assignment No. 5 (25 pts.) **Due Date: November 8, 2023 (Wednesday)**

1. (7 pts.) Find the point spectrum (eigenvalues) of the projection operator P_G, where G is the proper subspace of the Hilbert space $\mathcal{H}$ (i.e. $G \subset \mathcal{H}$, $G \neq \mathcal{H}$, $G \neq \{\Theta\}$, where Θ is the zero vector). Find the corresponding eigenspaces (sets of eigenvectors corresponding to eigenvalues of P_G).

2. (9 pts.) The operator U is called *unitary* if (i) its domain is the entire Hilbert space ($D(U) = \mathcal{H}$) (ii) its range is the entire Hilbert space ($\Delta(U) = \mathcal{H}$), and (iii) for every $f, g \in \mathcal{H}$,
$$(Uf, Ug) = (f, g). \tag{1}$$
   (a) Show that $U^{-1} = U^{\dagger}$.

   (b) The above definition of a unitary operator does not mention that unitary operators are also linear operators, and yet it can be shown that they are linear. In particular,
$$U(\alpha f) = \alpha U f, \tag{2}$$
   where α is a scalar (real or complex number). Use the inner product expression $(U(\alpha f) - \alpha U f, U(\alpha f) - \alpha U f)$, the bilinearity and positive-definitiness of the inner product, and the property represented by Eq. (1) to show that Eq. (2) is indeed true.

   (c) Show that eigenvectors f_1 and f_2 belonging to two different eigenvalues λ_1 and λ_2 of a unitary operator U are orthogonal.
   Hint: Consider (Uf_1, Uf_2), use the fact that each eigenvalue of a unitary operator U has an absolute value of 1 [see Problem 3 (b) in Assignment No. 4], and note that a complex number λ, whose absolute value is 1, can always be written as $\lambda = e^{i\alpha}$, where $0 \leq \alpha < 2\pi$.

3. (9 pts.) A linear operator A is called *skew-symmetric* if $A^{\dagger} = -A$.

   (a) Prove that the point spectrum (a set of eigenvalues) of the skew-symmetric operator is a subset of the imaginary axis.

   (b) Prove that the eigenvectors of a skew-symmetric operator A that correspond to two different eigenvalues of A are orthogonal.

   (c) Prove that a differential operator
$$\frac{\partial}{\partial \phi}$$
   is skew-symmetric (ϕ is a polar angle, $\phi \in [0, 2\pi]$).