

1. (8 pts.) The x , y , and z components of the angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} = -i\hbar(\hat{\mathbf{r}} \times \nabla)$ satisfy the following commutation relationships:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad (1)$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \quad (2)$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y. \quad (3)$$

Use the formulas

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad (4)$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad (5)$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \quad (6)$$

and the values of the elementary commutators involving the coordinate and momentum operators ($[\hat{x}, \hat{p}_x] = i\hbar\mathbf{1}$, $[\hat{x}, \hat{p}_y] = 0$, etc.) to prove Eq. (1). Then, use Eqs. (1)–(3) and algebraic properties of the commutator to show that $[\hat{L}^2, \hat{L}_z] = 0$, where $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.

Hint: Use the following properties of the commutator: $[\hat{F}_1\hat{F}_2, \hat{G}] = \hat{F}_1[\hat{F}_2, \hat{G}] + [\hat{F}_1, \hat{G}]\hat{F}_2$ and $[\hat{F}_1\hat{F}_2, \hat{G}_1\hat{G}_2] = \hat{F}_1\hat{G}_1[\hat{F}_2, \hat{G}_2] + \hat{F}_1[\hat{F}_2, \hat{G}_1]\hat{G}_2 + \hat{G}_1[\hat{F}_1, \hat{G}_2]\hat{F}_2 + [\hat{F}_1, \hat{G}_1]\hat{G}_2\hat{F}_2$.

2. (6 pts.) Consider an operator

$$\hat{A} = \alpha \frac{\partial}{\partial x}, \quad (7)$$

where α is a complex number.

- Find $\hat{A}^\dagger$ (the adjoint of $\hat{A}$).
 - Find values of α for which $\hat{A}$ becomes a Hermitian operator. Compare the result with an operator representing the x -component of the momentum $\mathbf{p}$. Please comment (briefly).
3. (6 pts.) Let A be the bounded linear operator whose domain is the entire Hilbert space $\mathcal{H}$ (so that you do not have to worry about the importance of domain in your calculations).
- Show that $A^{\dagger\dagger} = A$ [$A^{\dagger\dagger}$ is defined as $(A^\dagger)^\dagger$].
 - Assume that operator A is unitary, so that $(Af, Ag) = (f, g)$ for every $f, g \in \mathcal{H}$. Show that each eigenvalue of the unitary operator A has an absolute value equal 1 (one).

4. (5 pts.) Use the equation

$$[\hat{x}, \hat{H}] = \frac{i\hbar}{m} \hat{p}_x \quad (8)$$

(cf. Problem 4 of Assignment No. 3) to prove that the expectation value of the momentum operator $\hat{p}_x$ in the state $u = u(x)$, which is an eigenstate of the Hamiltonian $\hat{H} = \frac{\hat{p}_x^2}{2m} + \hat{V}(x)$, where $V(x)$ is an arbitrary potential function, is zero.

SOLUTIONS

$$\begin{aligned}
 1. \quad [\hat{L}_x, \hat{L}_y] &= [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] \\
 &= [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] - [\hat{y}\hat{p}_z, \hat{x}\hat{p}_z] \\
 &\quad - [\hat{z}\hat{p}_y, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z]
 \end{aligned}$$

Using the properties in the Hint, we obtain

$$\begin{aligned}
 [\overset{F_1}{\hat{y}} \overset{F_2}{\hat{p}_z}, \overset{G_1}{\hat{z}} \overset{G_2}{\hat{p}_x}] &= \overset{0}{\hat{y}\hat{z}[\hat{p}_z, \hat{p}_x]} + \overset{0}{\hat{y}[\hat{p}_z, \hat{z}]\hat{p}_x} \\
 &\quad + \overset{0}{\hat{z}[\hat{y}, \hat{p}_x]\hat{p}_z} + \overset{0}{[\hat{y}, \hat{z}]\hat{p}_x\hat{p}_z} \\
 &= -\hat{y} \underbrace{[\hat{z}, \hat{p}_z]}_{i\hbar} \hat{p}_x = -i\hbar \hat{y} \hat{p}_x
 \end{aligned}$$

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$$[\hat{y}_{F_1 F_2} \hat{p}_z, \hat{x}_{G_1 G_2} \hat{p}_z] = \hat{y} \hat{x} [\hat{p}_z, \hat{p}_z] \rightarrow 0$$

$$+ \hat{y} [\hat{p}_z, \hat{x}] \hat{p}_z + \hat{x} [\hat{y}, \hat{p}_z] \hat{p}_z \rightarrow 0$$

$$+ [\hat{y}, \hat{x}] \hat{p}_z^2 = 0$$

$$[\hat{z}_{F_1 F_2} \hat{p}_y, \hat{z}_{G_1 G_2} \hat{p}_x] = \hat{z}^2 [\hat{p}_y, \hat{p}_x] + \hat{z} [\hat{p}_y, \hat{z}] \hat{p}_x \rightarrow 0$$

$$+ \hat{z} [\hat{z}, \hat{p}_x] \hat{p}_y + [\hat{z}, \hat{z}] \hat{p}_x \hat{p}_y = 0$$

$$[\hat{z}_{F_1 F_2} \hat{p}_y, \hat{x}_{G_1 G_2} \hat{p}_z] = \hat{z} \hat{x} [\hat{p}_y, \hat{p}_z] + \hat{z} [\hat{p}_y, \hat{x}] \hat{p}_z \rightarrow 0$$

$$+ \hat{x} [\hat{z}, \hat{p}_z] \hat{p}_y + [\hat{z}, \hat{x}] \hat{p}_z \hat{p}_y \rightarrow 0$$

$$= \hat{x} \underbrace{[\hat{z}, \hat{p}_z]}_{i\hbar} \hat{p}_y = i\hbar \hat{x} \hat{p}_y$$

Thus,

$$[\hat{L}_x, \hat{L}_y] = -i\hbar \hat{y} \hat{p}_x + i\hbar \hat{x} \hat{p}_y$$

$$= i\hbar (\hat{x} \hat{p}_y - \hat{y} \hat{p}_x) = \underline{\underline{i\hbar \hat{L}_z}}$$

We now calculate $[\hat{L}^2, \hat{L}_z]$. Using the properties of the commutator mentioned in

the hint, we obtain,

$$[\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_z]$$

$$= [\overset{\text{F}_1 \text{F}_2}{\hat{L}_x^2}, \overset{\text{G}}{\hat{L}_z}] + [\overset{\text{F}_1 \text{F}_2}{\hat{L}_y^2}, \overset{\text{G}}{\hat{L}_z}]$$

$$+ \cancel{[\hat{L}_z^2, \hat{L}_z]} \overset{\rightarrow 0}{=} \hat{L}_x [\hat{L}_x, \hat{L}_z]$$

$$+ [\hat{L}_x, \hat{L}_z] \hat{L}_x + \hat{L}_y [\hat{L}_y, \hat{L}_z]$$

$$+ [\hat{L}_y, \hat{L}_z] \hat{L}_y.$$

Using Eqs. (2) and (3), we obtain,

$$[\hat{L}^2, \hat{L}_z] = -\hat{L}_x [\hat{L}_z, \hat{L}_x] - [\hat{L}_z, \hat{L}_x] \hat{L}_x$$

$$+ \hat{L}_y [\hat{L}_y, \hat{L}_z] + [\hat{L}_y, \hat{L}_z] \hat{L}_y$$

$$= -i\hbar \hat{L}_x \hat{L}_y - i\hbar \hat{L}_y \hat{L}_x$$

$$+ i\hbar \hat{L}_y \hat{L}_x + i\hbar \hat{L}_x \hat{L}_y$$

$$= 0.$$

$$\text{Thus, } [\hat{L}^2, \hat{L}_z] = 0,$$

which is what we had to show.

2. (a) We know that

$$\hat{A}^\dagger = \left(\alpha \frac{\partial}{\partial x} \right)^\dagger = \alpha^* \left(\frac{\partial}{\partial x} \right)^\dagger.$$

Let us find out what is $\left(\frac{\partial}{\partial x} \right)^\dagger$.

In general, the operator $\hat{F}^\dagger$ is defined by the condition

$$\int_{\Gamma} \varphi^* (\hat{F} \psi) d\tau = \int_{\Gamma} (\hat{F}^\dagger \varphi)^* \psi d\tau.$$

In our case, $\hat{F} = \frac{\partial}{\partial x}$, so that

$$\begin{aligned} & \int_{\mathbb{R}^3} \varphi^* \left(\frac{\partial}{\partial x} \psi \right) dx dy dz = \\ &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \varphi^* \frac{\partial \psi}{\partial x} \quad \begin{array}{l} \text{integrating} \\ \text{by} \\ \text{parts} \end{array} \\ &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \left\{ \underbrace{\varphi^* \psi}_{=0} \Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} \left(\frac{\partial \varphi}{\partial x} \right)^* \psi dx \right\} \end{aligned}$$

$$= - \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \left(\frac{\partial \varphi}{\partial x} \right)^* \psi$$

$$= \int_{\mathbb{R}^3} \left(- \frac{\partial}{\partial x} \varphi \right)^* \psi \, d\tau = \int_{\mathbb{R}^3} \left[\left(\frac{\partial}{\partial x} \right)^\dagger \varphi \right]^* \psi \, d\tau.$$

Thus, $\left(\frac{\partial}{\partial x} \right)^\dagger = - \frac{\partial}{\partial x},$

which means that

$$\hat{A}^\dagger = -\alpha^* \frac{\partial}{\partial x}.$$

(b) From the condition $\hat{A}^\dagger = A,$

we obtain

$$\alpha \frac{\partial}{\partial x} = -\alpha^* \frac{\partial}{\partial x},$$

so that $\hat{A}$ is Hermitian if and only if (iff)

$$\alpha = -\alpha^*.$$

If $\alpha = a + bi$ ($a = \text{Re}\alpha$, $b = \text{Im}\alpha$),

then

$$a + bi = -(a - bi) = -a + bi,$$

which implies that $a = 0$. Thus,

$\hat{A}$ is Hermitian iff

$$\alpha = bi, \text{ where } b \text{ is a real}$$

number.

You know that

$$\hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x} = -i\hbar \frac{\partial}{\partial x}.$$

Clearly, $\hat{p}_x$ is Hermitian, since in this case

we can write, $\hat{p}_x = \alpha \frac{\partial}{\partial x}$, $\alpha = -\hbar i$
($\hbar$ is real).

Moreover, if we are interested in

finding operators that might represent

$\hat{p}_x$ and that have the form

$$\propto \frac{\partial}{\partial x},$$

we ^{can} immediately narrow our search to

operators of the form

$$ib \frac{\partial}{\partial x}, \quad b - \text{real},$$

since otherwise we would have to deal

with operators that are not Hermitian.

(from this point of view, $b = -\frac{1}{2}$ is a constant
that guarantees the Hermiticity of $\hat{p}_x$).

3. (a) Consider turn-over-rule

$$(A^H f, g) = ((A^H)^H f, g)$$

$$= (f, A^H g) = (A^H g, f)^*$$

$$= (g, A f)^* = (A f, g)$$

Thus, $(A^H f, g) = (A f, g) \quad \forall f, g \in \mathcal{H}$

and, in consequence,

$$A^H f = A f \quad \forall f \in \mathcal{H}$$

$$\underline{\underline{A^H = A}}$$

(b) Consider

$$(Af, Af) = \|Af\|^2. \text{ For a unitary } A,$$

$$(Af, Af) = (f, f) = \|f\|^2.$$

Thus, $\|Af\| = \|f\|. \quad (\star)$

Assume that f is an eigenvector of A ,

$$Af = \lambda f, \quad \lambda - \text{eigenvalue.}$$

We obtain,

$$\|Af\| = \|\lambda f\| = |\lambda| \|f\| \stackrel{\text{Eq.}}{=} \underbrace{(\star)}_{\star} \|f\|.$$

Thus,

$$|\lambda| \|f\| = \|f\|. \text{ Since } f \neq \mathbf{0}$$

(f is an eigenvector, cannot be $\mathbf{0}$), $|\lambda| = 1$.

4. We have to calculate,

$$\langle \hat{p}_x \rangle \equiv \int_{-\infty}^{\infty} u^*(x) \hat{p}_x u(x) dx = (u, \hat{p}_x u),$$

where

$$\hat{H} u = E u.$$

According to the results of Problem 4 of Assignment No. 3, we can write,

$$[\hat{x}, \hat{H}] = \frac{i\hbar}{m} \hat{p}_x, \text{ or}$$

$$\hat{p}_x = \frac{m}{i\hbar} [\hat{x}, \hat{H}].$$

Thus,

$$\begin{aligned} \langle \hat{p}_x \rangle &= (u, \hat{p}_x u) = \\ &= \frac{m}{i\hbar} (u, [\hat{x}, \hat{H}] u) = \\ &= \frac{m}{i\hbar} (u, (\hat{x} \hat{H} - \hat{H} \hat{x}) u) = \\ &= \frac{m}{i\hbar} \{ (u, \hat{x} \hat{H} u) - (u, \hat{H} \hat{x} u) \} \end{aligned}$$

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$$\hat{H}u = Eu$$

$\hat{H}$ - hermitian

$$\frac{m}{i\hbar} \{ (u, E \hat{x} u) - (\hat{H} u, \hat{x} u) \}$$

$$\hat{H}u = Eu$$

E - real

$$\frac{m}{i\hbar} \{ E (u, \hat{x} u) - E (u, \hat{x} u) \}$$

$$= 0,$$

where we used the fact that $\hat{H}$ is hermitian
(turn-over rule) and the fact that E is
a real number.

Pish Sheikh