CEM 991 **Fall 2023**

Assignment No. 4 (25 pts.) **Due Date: October 27, 2023 (Friday)**

1. (8 pts.) The x, y, and z components of the angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} = -i\hbar(\hat{\mathbf{r}} \times \boldsymbol{\nabla})$ satisfy the following commutation relationships:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \tag{1}$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \tag{2}$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y. \tag{3}$$

Use the formulas

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \tag{4}$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \tag{5}$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \tag{6}$$

and the values of the elementary commutators involving the coordinate and momentum operators ($[\hat{x}, \hat{p}_x] = \imath\hbar\mathbf{1}$, $[\hat{x}, \hat{p}_y] = 0$, etc.) to prove Eq. (1). Then, use Eqs. (1)–(3) and algebraic properties of the commutator to show that $[\hat{L}^2, \hat{L}_z] = 0$, where $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.
Hint: Use the following properties of the commutator: $[\hat{F}_1\hat{F}_2, \hat{G}] = \hat{F}_1[\hat{F}_2, \hat{G}] + [\hat{F}_1, \hat{G}]\hat{F}_2$ and $[\hat{F}_1\hat{F}_2, \hat{G}_1\hat{G}_2] = \hat{F}_1\hat{G}_1[\hat{F}_2, \hat{G}_2] + \hat{F}_1[\hat{F}_2, \hat{G}_1]\hat{G}_2 + \hat{G}_1[\hat{F}_1, \hat{G}_2]\hat{F}_2 + [\hat{F}_1, \hat{G}_1]\hat{G}_2\hat{F}_2$.

2. (6 pts.) Consider an operator

$$\hat{A} = \alpha\frac{\partial}{\partial x}, \tag{7}$$

where α is a complex number.

   (a) Find $\hat{A}^\dagger$ (the adjoint of $\hat{A}$).

   (b) Find values of α for which $\hat{A}$ becomes a Hermitian operator. Compare the result with an operator representing the x-component of the momentum $\mathbf{p}$. Please comment (briefly).

3. (6 pts.) Let A be the bounded linear operator whose domain is the entire Hilbert space $\mathscr{H}$ (so that you do not have to worry about the importance of domain in your calculations).

   (a) Show that $A^{\dagger\dagger} = A$ [$A^{\dagger\dagger}$ is defined as $(A^\dagger)^\dagger$].

   (b) Assume that operator A is unitary, so that $(Af, Ag) = (f, g)$ for every f, $g \in \mathscr{H}$. Show that each eigenvalue of the unitary operator A has an absolute value equal 1 (one).

4. (5 pts.) Use the equation

$$[\hat{x}, \hat{H}] = \frac{i\hbar}{m}\,\hat{p}_x \tag{8}$$

(cf. Problem 4 of Assignment No. 3) to prove that the expectation value of the momentum operator $\hat{p}_x$ in the state $u = u(x)$, which is an eigenstate of the Hamiltonian $\hat{H} = \frac{\hat{p}_x^2}{2m} + \hat{V}(x)$, where $V(x)$ is an arbitrary potential function, is zero.