

1. (3 pts.) Let q_ℓ and p_ℓ , $\ell = 1, \dots, f$, designate the generalized coordinates and conjugate momenta, respectively, and let $\{ , \}$ be the Poisson bracket. Prove the following relationships:

(a) $\{q_\ell, q_{\ell'}\} = 0$,

(b) $\{p_\ell, p_{\ell'}\} = 0$,

(c) $\{q_\ell, p_{\ell'}\} = \delta_{\ell\ell'}$.

2. (5 pts.) Let F_i , G_i , $i = 1, 2$, and G be functions of $q_1, \dots, q_f, p_1, \dots, p_f$, and t and let $\{ , \}$ designate the Poisson bracket. Prove the following properties of the Poisson bracket:

(a) $\{F_1, F_2\} = -\{F_2, F_1\}$,

(b) $\{F_1 F_2, G\} = F_1 \{F_2, G\} + \{F_1, G\} F_2$,

(c) $\{F_1 F_2, G_1 G_2\} = F_1 G_1 \{F_2, G_2\} + F_1 \{F_2, G_1\} G_2 + G_1 \{F_1, G_2\} F_2 + \{F_1, G_1\} G_2 F_2$.

Hint: To prove (c), use (a) and (b) replacing G by $G_1 G_2$.

3. (5 pts.) It can be shown that the Poisson brackets have the following two properties:

(a) $\{\{F_1, F_2\}, F_3\} + \{\{F_2, F_3\}, F_1\} + \{\{F_3, F_1\}, F_2\} = 0$ (Jacobi's identity),

(b) $\frac{\partial}{\partial t} \{F_1, F_2\} = \left\{ \frac{\partial}{\partial t} F_1, F_2 \right\} + \left\{ F_1, \frac{\partial}{\partial t} F_2 \right\}$,

where F_i , $i = 1 - 3$, are functions of $q = (q_1, \dots, q_f)$, $p = (p_1, \dots, p_f)$, and t . Prove property (b). Use properties (a) and (b) to prove the following theorem (known as the Poisson-Jacobi theorem): *If two dynamical variables F_1 and F_2 are constants of motion, i.e., $F_1(q, p, t) = \text{const}$ and $F_2(q, p, t) = \text{const}$, then the Poisson bracket of F_1 and F_2 , i.e., $\{F_1, F_2\}$, is also a constant of motion.*

Hint: Use the equation of motion in terms of the Poisson brackets for the dynamical variable $F = \{F_1, F_2\}$ and use property (a) (Jacobi's identity) with $F_3 = H$, where H is the Hamilton function. Next, use the equation of motion in terms of the Poisson brackets for dynamical variables F_1 and F_2 , knowing that $dF_1/dt = 0$ and $dF_2/dt = 0$. Finally, use property (b) to demonstrate that $(d/dt)\{F_1, F_2\} = 0$.

4. (5 pts.) Use the results of Problems 1 and 2 to demonstrate that

$$\{L_x, L_y\} = L_z, \quad (1)$$

$$\{L_y, L_z\} = L_x, \quad (2)$$

$$\{L_z, L_x\} = L_y, \quad (3)$$

where L_x , L_y , and L_z are the x , y , and z components of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}^{\#}$, respectively, and $\{ , \}$ is the Poisson bracket. It is enough if you derive Eq. (1), since the remaining two equations, Eqs. (2) and (3), can be obtained by cyclic permutations of variables x , y , and z . Use the above Eqs. (1)–(3) and the Poisson-Jacobi theorem formulated in Problem 3 to show that if any two components of the angular momentum are constant during motion, then the third one must be constant as well.

5. (7 pts.) Prove that relationships (a)–(c) listed in Problem 2 remain valid if we replace F_i , G_i , $i = 1, 2$, and G by linear operators and the Poisson brackets $\{ , \}$ by commutators $[,]$. Prove that Jacobi's identity (relationship (a) in Problem 3) remains valid if we replace F_i , $i = 1 - 3$, by linear operators and the Poisson brackets $\{ , \}$ by commutators $[,]$.

^{#)} Bold symbols designate vector quantities.