CEM 991 Fall 2023

Assignment No. 1 (25 pts.) Due Date: September 20, 2023 (Wednesday)

1. (4 pts.) Consider the classical particle with mass m. Let x, y, and z define the Cartesian components of the radius vector[#] $\mathbf{r}$ from the origin to the particle and let r, θ, and ϕ designate the spherical components of $\mathbf{r}$, so that

$$x = r\sin\theta\,\cos\phi, \tag{1}$$
$$y = r\sin\theta\,\sin\phi, \tag{2}$$
$$z = r\cos\theta. \tag{3}$$

Derive the formula for the kinetic energy of the particle m,

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2). \tag{4}$$

Recall that the dot above a symbol designates the time derivative $\frac{d}{dt}$.

2. (8 pts.) Consider the classical particle with mass m under the influence of a force derived from the potential $V(\mathbf{r}, t)$ (no constraints). You have learned in class that the canonical momenta p_ℓ, $\ell = 1, 2, 3$, are related to Cartesian components of the linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ as follows:

$$p_\ell = p_x\frac{\partial x}{\partial q_\ell} + p_y\frac{\partial y}{\partial q_\ell} + p_z\frac{\partial z}{\partial q_\ell}. \tag{5}$$

Let r, θ, and ϕ designate the spherical components of $\mathbf{r}$, so that $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$.

(a) Using the general definition of the canonical momentum p_ℓ,

$$p_\ell = \frac{\partial L}{\partial \dot{q}_\ell}, \quad L = T - V, \tag{6}$$

and the fact that the kinetic energy T expressed in terms of r, θ, and ϕ is given by Eq. (4) (see Problem 1), prove that

$$p_r = m\dot{r}, \tag{7}$$
$$p_\theta = mr^2\dot{\theta}, \tag{8}$$
$$p_\phi = mr^2\sin^2\theta\,\dot{\phi}. \tag{9}$$

(b) Show that

$$p_r = \mathbf{p}\bullet\frac{\mathbf{r}}{r}, \tag{10}$$

so that p_r has an interpretation of a linear momentum along the radial direction.

[#]) Bold symbols designate vector quantities.

(c) Show that p_ϕ can be interpreted as the z-component of the angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{11}$$

In other words, L_z is conjugate to ϕ. *Hint*: Note that

$$L_z = xp_y - yp_x = m(x\dot{y} - y\dot{x}), \tag{12}$$

and use the relationships (1)–(3) and the chain rule to express $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$.

(d) By using the equations presented in (a) and (b) and the definition of the Hamiltonian,

$$H = \sum_{\ell=1}^{3} p_\ell v_\ell - L, \tag{13}$$

where $L = T - V$ is the Langrangian and $v_\ell = \dot{q}_\ell$ (recall that v_ℓ has to be expressed in terms of q_ℓ and p_ℓ, $\ell = 1, 2, 3$), show that

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V. \tag{14}$$

Show that the same expression for H is obtained by using the formula

$$H = T + V. \tag{15}$$

Explain why Eqs. (13) and (15) lead to the same result.

3. (8 pts.) The force that governs the motion of a harmonic oscillator with mass m and the angular frequency ω is defined as

$$F_x = -kx, \tag{16}$$

where $k = m\omega^2$.

(a) Calculate the potential function V [assume that $V(x = 0) = 0$].

(b) Calculate the Lagrangian and the Hamiltonian.

(c) Show that the Lagrange equation reduces in this case to ordinary Newton's equation

$$m\ddot{x} = F_x, \tag{17}$$

where F_x is defined by Eq. (16).

(d) Solve Newton's equation [Eq. (17)], i.e. determine $x = x(t)$, assuming the following initial conditions: $x(t = 0) = 0$, $\dot{x}(t = 0) = v_0$. Use the solution $x(t)$, the definition of the kinetic energy T, and the formula for the potential function V [see item (a) above] to show that

$$T + V = \text{const.} \tag{18}$$

What is the value of $T + V$ in this case?

4. (5 pts.) The force on a particle with charge q is given by the well-known Lorentz formula

$$\mathbf{F} = q\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})\right], \tag{19}$$

where

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{20}$$

and

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{21}$$

are, respectively, the electric and magnetic field vectors, ϕ is the electric potential, $\mathbf{A}$ is the magnetic vector potential, and $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ is the velocity. The velocity-dependent potential U corresponding to Lorentz force $\mathbf{F}$, Eq. (19), is given by the formula,

$$U = q\phi - \frac{q}{c}\mathbf{A}\bullet\mathbf{v}. \tag{22}$$

Show that the following equation,

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}, \tag{23}$$

is true.
Hint: Calculate $\dfrac{\partial U}{\partial x}$ and $\dfrac{\partial U}{\partial \dot{x}}$ using formula (22) and compare the result with the formula for the x component of $\mathbf{F}$ resulting from Eqs. (19)–(21). Recall that for any vector field $\mathbf{G} = \mathbf{G}(x, y, z, t)$,

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\mathbf{G})_x &= \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}, && (24)\\
(\boldsymbol{\nabla}\times\mathbf{G})_y &= \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}, && (25)\\
(\boldsymbol{\nabla}\times\mathbf{G})_z &= \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}. && (26)
\end{aligned}
$$

SOLUTIONS

1. The kinetic energy is defined as follows,

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right).$$

From Eqs. (1) – (3) it immediately follows that

$$\dot{x} = \dot{r} \sin\Theta \cos\varphi + r \cos\Theta\, \dot{\Theta} \cos\varphi - r \sin\Theta \sin\varphi\, \dot{\varphi},$$

$$\dot{y} = \dot{r} \sin\Theta \sin\varphi + r \cos\Theta\, \dot{\Theta} \sin\varphi + r \sin\Theta \cos\varphi\, \dot{\varphi},$$

$$\dot{z} = \dot{r} \cos\Theta - r \sin\Theta\, \dot{\Theta}$$

Thus, we obtain

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \underline{\dot{r}^2 \sin^2\Theta \cos^2\varphi} + \underline{\underline{r^2 \cos^2\Theta \cos^2\varphi\, \dot{\Theta}^2}} + \underline{\underline{\underline{r^2 \sin^2\Theta \sin^2\varphi\, \dot{\varphi}^2}}}$$

$$
\begin{aligned}
&+ 2 r \dot{r} \dot{\theta} \sin\theta \cos\theta \cos^2\varphi \\
&\cancel{- 2 r \dot{r} \dot{\varphi} \sin^2\theta \sin\varphi \cos\varphi} \\
&\cancel{- 2 r^2 \sin\theta \cos\theta \sin\varphi \cos\varphi \, \dot{\theta} \dot{\varphi}} \\
&+ \dot{r}^2 \sin^2\theta \sin^2\varphi + r^2 \cos^2\theta \sin^2\varphi \, \dot{\theta}^2 \\
&+ r^2 \sin^2\theta \cos^2\varphi \, \dot{\varphi}^2 \\
&+ 2 r \dot{r} \dot{\theta} \sin\theta \cos\theta \sin^2\varphi \\
&\cancel{+ 2 r \dot{r} \dot{\varphi} \sin^2\theta \sin\varphi \cos\varphi} \\
&\cancel{+ 2 r^2 \sin\theta \cos\theta \sin\varphi \cos\varphi \, \dot{\theta} \dot{\varphi}} \\
&+ \dot{r}^2 \cos^2\theta + r^2 \sin^2\theta \, \dot{\theta}^2 \\
&- 2 r \dot{r} \dot{\theta} \sin\theta \cos\theta \, . \qquad (1.1)
\end{aligned}
$$

By using the well-known formula

$$\sin^2\alpha + \cos^2\alpha = 1, \qquad (1.2)$$

we can simplify Eq. (1.1) as follows,

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta$$
$$+ r^2\dot{\varphi}^2\sin^2\theta + \cancel{2r\dot{r}\dot{\theta}\sin\theta\cos\theta}$$
$$+ \dot{r}^2\cos^2\theta + r^2\sin^2\theta\,\dot{\theta}^2$$
$$- \cancel{2r\dot{r}\dot{\theta}\sin\theta\cos\theta}$$
$$\overset{\text{Eq.(1.2)}}{=} \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2.$$

As a result,

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right).$$

2. (a) Since V depends on $\vec{r}$ and t and does not depend on $\dot{\vec{r}}$, we have

$$\frac{\partial V}{\partial \dot{q}_l} = 0.$$

In consequence,

$$p_l = \frac{\partial L}{\partial \dot{q}_l} = \frac{\partial T}{\partial \dot{q}_l}.$$

In particular,

$$p_r = \frac{\partial T}{\partial \dot{r}}, \quad p_\theta = \frac{\partial T}{\partial \dot{\theta}}, \quad p_\varphi = \frac{\partial T}{\partial \dot{\varphi}}.$$

From Eq. (4) (cf. Problem 1), we obtain

$$p_r = \frac{\partial T}{\partial \dot{r}} = \frac{1}{2} m \, 2\dot{r} = m\dot{r},$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2} m r^2 \, 2\dot{\theta} = m r^2 \dot{\theta},$$

$$p_\varphi = \frac{\partial T}{\partial \dot{\varphi}} = \frac{1}{2} m r^2 \sin^2\theta \, 2\dot{\varphi} = m r^2 \sin^2\theta \, \dot{\varphi}.$$

(b) From Eq. (5), we obtain

$$p_r = p_x \frac{\partial x}{\partial r} + p_y \frac{\partial y}{\partial r} + p_z \frac{\partial z}{\partial r}.$$

Now,

$$\frac{\partial x}{\partial r} = \sin\theta \cos\varphi = \frac{x}{r},$$

$$\frac{\partial y}{\partial r} = \sin\theta \sin\varphi = \frac{y}{r},$$

$$\frac{\partial z}{\partial r} = \cos\theta = \frac{z}{r}.$$

This means that

$$p_r = p_x \frac{x}{r} + p_y \frac{y}{r} + p_z \frac{z}{r} =$$
$$= \vec{p} \cdot \frac{\vec{r}}{r}$$

(c) The z-component of $\vec{L} = \vec{r} \times \vec{p}$ equals

$$L_z = x p_y - y p_x = m(x\dot{y} - y\dot{x}).$$

We have already demonstrated (see the solution of Problem 1) that

$$\dot{x} = \dot{r} \sin\theta \cos\varphi + r \cos\theta \cos\varphi \,\dot{\theta}$$
$$- r \sin\theta \sin\varphi \,\dot{\varphi}$$

$$\dot{y} = \dot{r} \sin\theta \sin\varphi + r \cos\theta \sin\varphi \,\dot{\theta}$$
$$+ r \sin\theta \cos\varphi \,\dot{\varphi}$$

We also know that

$$x = r \sin\theta \cos\varphi,$$

$$y = r \sin\theta \sin\varphi.$$

From the above four equations, we obtain

$$\begin{aligned}
x\dot{y} - y\dot{x} &= \cancel{r\dot{r} \sin^2\theta \sin\varphi \cos\varphi} \\
&+ \cancel{r^2 \sin\theta \cos\theta \sin\varphi \cos\varphi\, \dot{\theta}} \\
&+ r^2 \sin^2\theta \cos^2\varphi\, \dot{\varphi} \\
&- \cancel{r\dot{r} \sin^2\theta \sin\varphi \cos\varphi} \\
&- \cancel{r^2 \sin\theta \cos\theta \sin\varphi \cos\varphi\, \dot{\theta}} \\
&+ r^2 \sin^2\theta \sin^2\varphi\, \dot{\varphi} \\
&= r^2 \sin^2\theta \left(\sin^2\varphi + \cos^2\varphi\right) \dot{\varphi} \\
&= r^2 \sin^2\theta\, \dot{\varphi}.
\end{aligned}$$

Thus,

$$L_z = m(x\dot{y} - y\dot{x}) = m r^2 \sin^2\theta\, \dot{\varphi}$$
$$= p_\varphi$$

(d) Let us begin with the original definition of H,

$$H = \sum_{l=1}^{3} p_l v_l - L =$$

$$= p_r \dot{r} + p_\theta \dot{\theta} + p_\varphi \dot{\varphi} - T + V. \quad (2.1)$$

From Eqs. (7) - (9), we obtain,

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{m r^2}, \quad \dot{\varphi} = \frac{p_\varphi}{m r^2 \sin^2\theta},$$

so that

$$p_r \dot{r} + p_\theta \dot{\theta} + p_\varphi \dot{\varphi} = \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} + \frac{p_\varphi^2}{m r^2 \sin^2\theta} = \frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta} \right) \tag{2.2}$$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 \right) = \frac{1}{2} m \left(\frac{p_r^2}{m^2} + r^2 \frac{p_\theta^2}{m^2 r^4} + r^2 \sin^2\theta \frac{p_\varphi^2}{m^2 r^4 \sin^4\theta} \right) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta} \right) \tag{2.3}$$

By combining Eqs. (2.1) – (2.3), we find that

$$H = \frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta} \right) - \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta} \right) + V$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + V \qquad (2.4)$$

From Eq. (14) and Eq. (2.3), we immediately obtain,

$$H = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right)$$
$$+ V,$$

which is identical to Eq. (2.4).

Eqs. (13) and (14) must give the same result, since

- Eqs. (1) – (3) defining the spherical coordinates do not depend on time $\left(\frac{\partial x}{\partial t} = \frac{\partial y}{\partial t} = \frac{\partial z}{\partial t} = 0\right)$

- We consider here a force that can be derived from the potential $V(\vec{r},t)$. In the class, we have learnt that if the above two conditions are satisfied, H equals T+V.

3. (a)

$$V(x) = V(x_0) - \int_{x_0}^{x} F_x(x')\,dx'$$

Let us choose $x_0 = 0$, so that we can write

$$V(x) = \underbrace{V(x=0)}_{=0} - \int_{0}^{x} F_x(x')\,dx'$$

$$= - \int_{0}^{x} F_x(x')\,dx' =$$

$$= +k \int^{x} x'\,dx'.$$

We obtain,

$$V(x) = k\left.\frac{(x')^2}{2}\right|_{x'=0}^{x'=x} =$$

$$= \frac{1}{2}kx^2.$$

$\therefore$

(b) $$L(x,\dot{x},t) = T(\dot{x}) - V(x,t)$$

$$T(\dot{x}) = \frac{1}{2}m\dot{x}^2$$

$$V(x,t) \equiv V(x) = \frac{1}{2}kx^2.$$

$$L = L(x,\dot{x},t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Since there are no constraints and a force can be derived from $V(x)$,

$$H = T + V.$$

We obtain,

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} \Rightarrow$$

$$\Rightarrow \quad \dot{x} = \frac{p_x}{m}.$$

Thus,

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \cdot \frac{p_x^2}{m^2} =$$

$$= \frac{1}{2m} p_x^2,$$

so that

$$H = H(x, p_x, t) = \frac{1}{2m} p_x^2 +$$

$$+ \frac{1}{2} k x^2$$

(c) The Lagrange equation has the form,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x},$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = -\frac{1}{\cancel{2}}k(\cancel{2}x) = -kx.$$

Thus,
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + kx = 0,$$

so that
$$m\ddot{x} = -kx = F_x$$

(d) For ODEs of the general form,
$$a\ddot{x} + b\dot{x} + cx = 0 \quad (a, b, c - \text{constants}),$$

we always first solve the characteristic equation,
$$a\lambda^2 + b\lambda + c = 0, \text{ and}$$

use roots $\lambda_{1,2}$ to construct the general solution. If both λ_1 and λ_2 are real and $\lambda_1 \neq \lambda_2$, we have

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \quad (A, B\text{-constants}).$$

If λ_1 and λ_2 are complex,

$$\lambda_{1,2} = \alpha \pm \beta i,$$

then

$$x(t) = e^{\alpha t}(A \cos \beta t + B \sin \beta t)$$

In our case, our ODE has the form,

$$m\ddot{x} + kx = 0.$$

The characteristic equation,

$$m\lambda^2 + k = 0,$$

has two roots,

$$\lambda_{1,2} = \pm\sqrt{\frac{k}{m}}\, i = \pm i\omega,$$

where ω is the angular frequency.

Thus, the general solution of Newton's equation is

$$x(t) = A\cos\omega t + B\sin\omega t,$$

$x(t=0)=0$ implies that

$$A = 0.$$

$$\dot{x}(t) = -\omega A\sin\omega t + B\omega\cos\omega t,$$

so that

$$\dot{x}(t=0) = B\omega = v_0,$$

and

$$B = \frac{v_0}{\omega},$$

In consequence,

$$x(t) = \frac{v_0}{\omega}\sin\omega t.$$

Now,

$$T+V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

$$\dot{x}(t) = \frac{v_0}{\cancel{\omega}}\cancel{\omega}\cos\omega t = v_0\cos\omega t$$

$$\frac{1}{2}m\dot{x}^2 = \frac{1}{2}mv_0^2\cos^2\omega t$$

$$\frac{1}{2}kx^2 = \frac{1}{2}k\frac{v_0^2}{\omega^2}\sin^2\omega t$$

$$= \frac{1}{2}m\cancel{\omega^2}\frac{v_0^2}{\cancel{\omega^2}}\sin^2\omega t$$

$$= \frac{1}{2}mv_0^2\sin^2\omega t.$$

Thus,

$$T+V = \frac{1}{2}mv_0^2\cos^2\omega t +$$

$$+\frac{1}{2}mv_0^2\sin^2\omega t = \frac{1}{2}mv_0^2.$$

T+V is a constant and the value of

$T+V$ is

$$\frac{1}{2} m v_0^2.$$

(For example,

- at $t=0$, $T=\frac{1}{2} m v_0^2$, $V=0$, $T+V=\frac{1}{2} m v_0^2$
- at $t=\frac{\pi}{2\omega}=\frac{\cancel{\pi}}{4\cancel{\pi}\nu}=\frac{T}{4}$ (ν - frequency, T - period),

$T=0$, $V=\frac{1}{2} m v_0^2$, $T+V=\frac{1}{2} m v_0^2$)

4. According to Eq. (22),

$$U = q\varphi - \frac{q}{c}\vec{A}\cdot\vec{v} =$$

$$= q\varphi - \frac{q}{c}\left(A_x\dot{x}+A_y\dot{y}+A_z\dot{z}\right).$$

Let us calculate $\frac{\partial U}{\partial \dot{x}}$ and $\frac{\partial U}{\partial x}$.

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} A_x \qquad (4.1)$$

(recall that φ and $\vec{A}$ depend on x, y, z, and t and that they do not depend on $\dot{x}$, $\dot{y}$, and $\dot{z}$),

$$\frac{\partial U}{\partial x} = q\,\frac{\partial \varphi}{\partial x} - \frac{q}{c}\left(\dot{x}\,\frac{\partial A_x}{\partial x} + \dot{y}\,\frac{\partial A_y}{\partial x} + \dot{z}\,\frac{\partial A_z}{\partial x}\right) \qquad (4.2)$$

From Eq. (4.1) we obtain,

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right) = -\frac{q}{c}\,\frac{dA_x}{dt} =$$

$$= -\frac{q}{c}\left(\frac{\partial A_x}{\partial x}\,\dot{x} + \frac{\partial A_x}{\partial y}\,\dot{y} + \frac{\partial A_x}{\partial z}\,\dot{z} + \frac{\partial A_x}{\partial t}\right) \qquad (4.3)$$

Thus,

$$-\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right) = -q\,\frac{\partial \varphi}{\partial x} + \frac{q}{c}\left(\dot{x}\,\frac{\partial A_x}{\partial x} + \dot{y}\,\frac{\partial A_y}{\partial x} + \dot{z}\,\frac{\partial A_z}{\partial x}\right)$$

$$-\frac{q}{c}\left(\dot{x}\frac{\partial A_x}{\partial x}+\dot{y}\frac{\partial A_x}{\partial y}+\dot{z}\frac{\partial A_x}{\partial z}+\frac{\partial A_x}{\partial t}\right)$$

$$=-q\frac{\partial \varphi}{\partial x}-\frac{q}{c}\left(\cancel{-\dot{x}\frac{\partial A_x}{\partial x}}-\dot{y}\frac{\partial A_y}{\partial x}\right.$$

$$-\dot{z}\frac{\partial A_z}{\partial x}+\cancel{\dot{x}\frac{\partial A_x}{\partial x}}+\dot{y}\frac{\partial A_x}{\partial y}+\dot{z}\frac{\partial A_x}{\partial z}$$

$$\left.+\frac{\partial A_x}{\partial t}\right)$$

$$=-q\frac{\partial \varphi}{\partial x}-\frac{q}{c}\left[-\dot{y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right.$$

$$\left.+\dot{z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial A_x}{\partial t}\right] \tag{4.4}$$

Recall that

$$\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=\left(\vec{\nabla}\times\vec{A}\right)_z ,$$

$$\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}=\left(\vec{\nabla}\times\vec{A}\right)_y ,$$

so that Eq. (4.4) becomes,

$$-\frac{\partial U}{\partial x}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)=q\left(-\frac{\partial \varphi}{\partial x}-\frac{1}{c}\frac{\partial A_x}{\partial t}\right)$$

$$-\frac{q}{c}\left[-\dot{y}(\vec{\nabla}\times\vec{A})_z+\dot{z}(\vec{\nabla}\times A)_y\right].$$

From Eqs. (20) and (21),

$$-\frac{\partial \varphi}{\partial x}-\frac{1}{c}\frac{\partial A_x}{\partial t}=E_x \quad \text{and}$$

$$\vec{\nabla}\times\vec{A}=\vec{B}, \quad \text{so that}$$

$$-\frac{\partial U}{\partial x}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)=qE_x+$$

$$+\frac{q}{c}\left(\dot{y}B_z-\dot{z}B_y\right)$$

$$=qE_x+\frac{q}{c}\left(\vec{v}\times\vec{B}\right)_x=F_x,$$

where F_x is the x-component of

$\vec{F}$, Eq. (19).

In consequence,

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right),$$

which is a desired result, Eq. (23)

$\therefore$

Note: Similar arguments would allow us to prove that Eq. (23) is also true for the y and z components of $\vec{F}$.