CEM 991 Fall 2023

Assignment No. 1 (25 pts.) Due Date: September 20, 2023 (Wednesday)

1. (4 pts.) Consider the classical particle with mass m. Let x, y, and z define the Cartesian components of the radius vector[#] $\mathbf{r}$ from the origin to the particle and let r, θ, and ϕ designate the spherical components of $\mathbf{r}$, so that

$$x = r\sin\theta\,\cos\phi, \tag{1}$$
$$y = r\sin\theta\,\sin\phi, \tag{2}$$
$$z = r\cos\theta. \tag{3}$$

Derive the formula for the kinetic energy of the particle m,

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2). \tag{4}$$

Recall that the dot above a symbol designates the time derivative $\frac{d}{dt}$.

2. (8 pts.) Consider the classical particle with mass m under the influence of a force derived from the potential $V(\mathbf{r}, t)$ (no constraints). You have learned in class that the canonical momenta p_ℓ, $\ell = 1, 2, 3$, are related to Cartesian components of the linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ as follows:

$$p_\ell = p_x\frac{\partial x}{\partial q_\ell} + p_y\frac{\partial y}{\partial q_\ell} + p_z\frac{\partial z}{\partial q_\ell}. \tag{5}$$

Let r, θ, and ϕ designate the spherical components of $\mathbf{r}$, so that $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$.

(a) Using the general definition of the canonical momentum p_ℓ,

$$p_\ell = \frac{\partial L}{\partial \dot{q}_\ell}, \quad L = T - V, \tag{6}$$

and the fact that the kinetic energy T expressed in terms of r, θ, and ϕ is given by Eq. (4) (see Problem 1), prove that

$$p_r = m\dot{r}, \tag{7}$$
$$p_\theta = mr^2\dot{\theta}, \tag{8}$$
$$p_\phi = mr^2\sin^2\theta\,\dot{\phi}. \tag{9}$$

(b) Show that

$$p_r = \mathbf{p}\bullet\frac{\mathbf{r}}{r}, \tag{10}$$

so that p_r has an interpretation of a linear momentum along the radial direction.

[#]) Bold symbols designate vector quantities.

(c) Show that p_ϕ can be interpreted as the z-component of the angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{11}$$

In other words, L_z is conjugate to ϕ. *Hint*: Note that

$$L_z = xp_y - yp_x = m(x\dot{y} - y\dot{x}), \tag{12}$$

and use the relationships (1)–(3) and the chain rule to express $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$.

(d) By using the equations presented in (a) and (b) and the definition of the Hamiltonian,

$$H = \sum_{\ell=1}^{3} p_\ell v_\ell - L, \tag{13}$$

where $L = T - V$ is the Langrangian and $v_\ell = \dot{q}_\ell$ (recall that v_ℓ has to be expressed in terms of q_ℓ and p_ℓ, $\ell = 1, 2, 3$), show that

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V. \tag{14}$$

Show that the same expression for H is obtained by using the formula

$$H = T + V. \tag{15}$$

Explain why Eqs. (13) and (15) lead to the same result.

3. (8 pts.) The force that governs the motion of a harmonic oscillator with mass m and the angular frequency ω is defined as

$$F_x = -kx, \tag{16}$$

where $k = m\omega^2$.

(a) Calculate the potential function V [assume that $V(x = 0) = 0$].

(b) Calculate the Lagrangian and the Hamiltonian.

(c) Show that the Lagrange equation reduces in this case to ordinary Newton's equation

$$m\ddot{x} = F_x, \tag{17}$$

where F_x is defined by Eq. (16).

(d) Solve Newton's equation [Eq. (17)], i.e. determine $x = x(t)$, assuming the following initial conditions: $x(t = 0) = 0$, $\dot{x}(t = 0) = v_0$. Use the solution $x(t)$, the definition of the kinetic energy T, and the formula for the potential function V [see item (a) above] to show that

$$T + V = \text{const}. \tag{18}$$

What is the value of $T + V$ in this case?

4. (5 pts.) The force on a particle with charge q is given by the well-known Lorentz formula

$$\mathbf{F} = q\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})\right], \tag{19}$$

where

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{20}$$

and

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{21}$$

are, respectively, the electric and magnetic field vectors, ϕ is the electric potential, $\mathbf{A}$ is the magnetic vector potential, and $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ is the velocity. The velocity-dependent potential U corresponding to Lorentz force $\mathbf{F}$, Eq. (19), is given by the formula,

$$U = q\phi - \frac{q}{c}\mathbf{A}\bullet\mathbf{v}. \tag{22}$$

Show that the following equation,

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}, \tag{23}$$

is true.

Hint: Calculate $\dfrac{\partial U}{\partial x}$ and $\dfrac{\partial U}{\partial \dot{x}}$ using formula (22) and compare the result with the formula for the x component of $\mathbf{F}$ resulting from Eqs. (19)–(21). Recall that for any vector field $\mathbf{G} = \mathbf{G}(x, y, z, t)$,

$$(\boldsymbol{\nabla}\times\mathbf{G})_x = \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}, \tag{24}$$

$$(\boldsymbol{\nabla}\times\mathbf{G})_y = \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}, \tag{25}$$

$$(\boldsymbol{\nabla}\times\mathbf{G})_z = \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}. \tag{26}$$