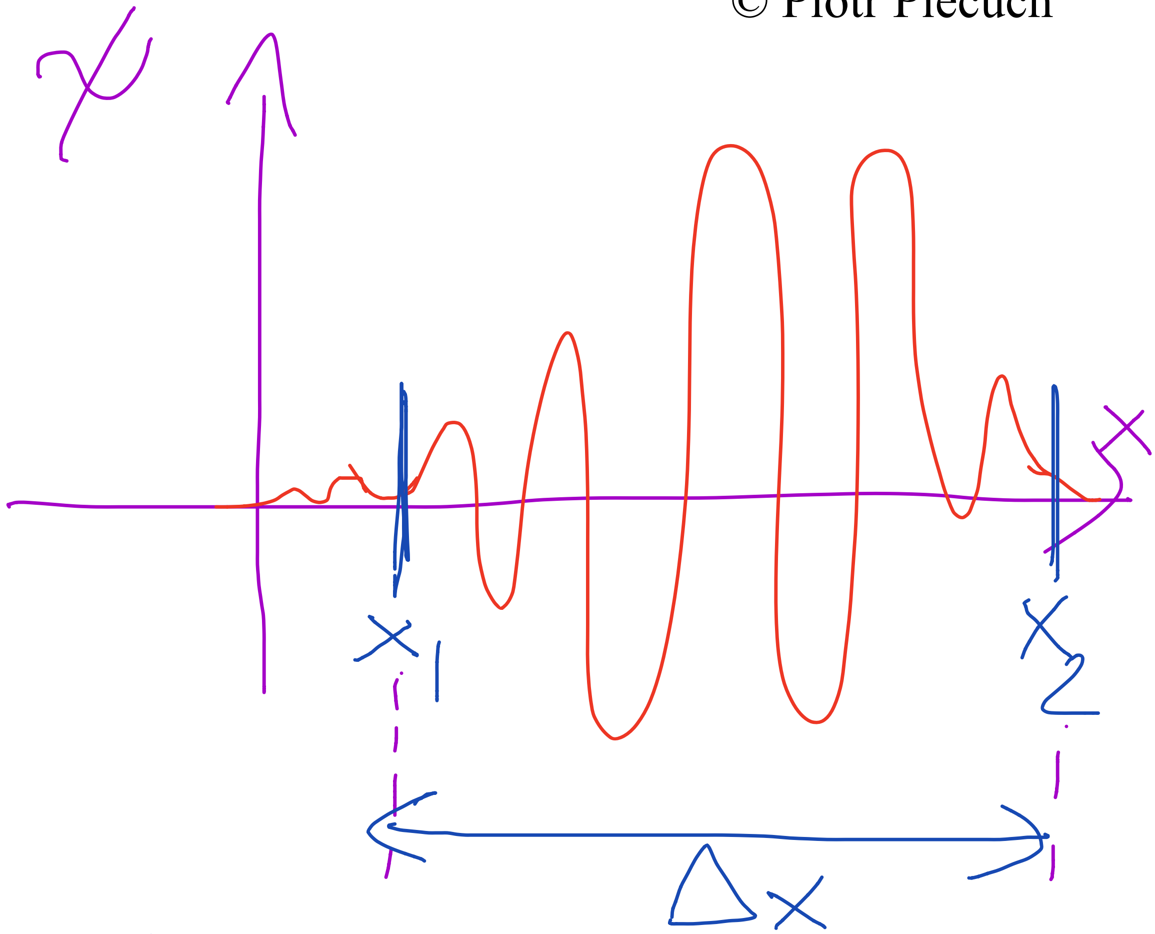


$$\Delta x = x_2 - x_1$$

$$\psi(x,t) = \sum_{\omega,k} A_{\omega,k} \times e^{i(kx-\omega t)}$$

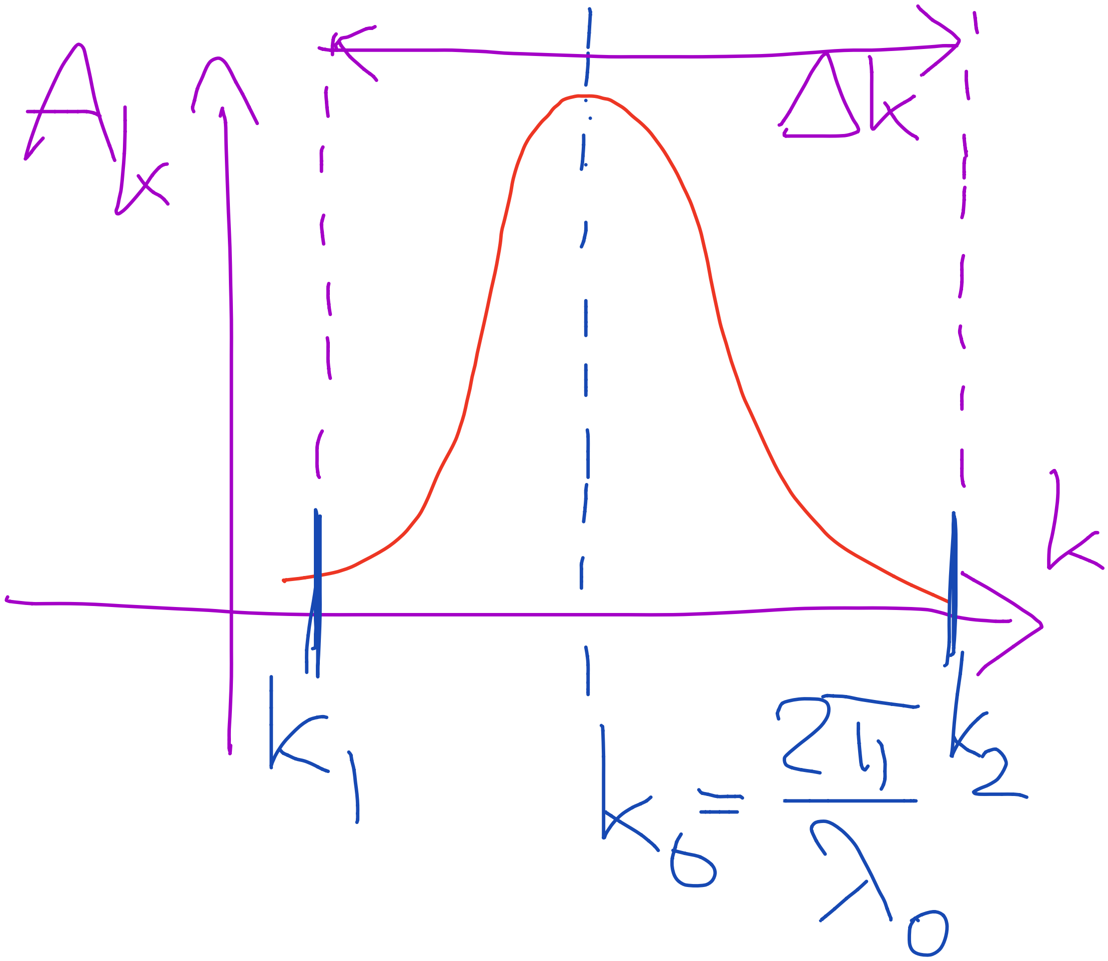
$$k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi\nu$$

Focus on x-dependence:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_k e^{ikx} dk$$

(Fourier transform)

$$A_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x)\, dx$$



$$\Delta k = k_2 - k_1$$

$$A_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x)\, dx$$

$$\checkmark \quad A_k \approx \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-ikx} \psi(x)\, dx$$

$$\checkmark \quad \underline{\psi(x)} \approx \frac{1}{\sqrt{2\pi}} \int_{k_1}^{k_2} \underline{e^{ikx} A_k}\, dk$$

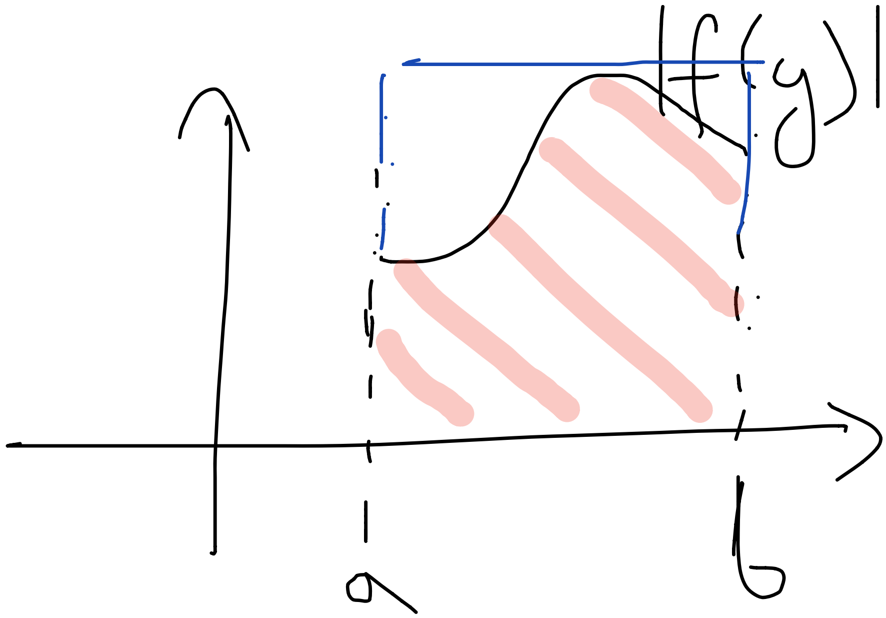
$$\left|\int_a^b f(y)\,dy\right| \leq \int_a^b |f(y)|\,dy$$

$$|\chi(x)| \leq \frac{1}{\sqrt{2\pi}} \int_{k_1}^{k_2} |A_k|\,dk$$

$$|e^{ikx}| = 1$$

$$\int_a^b |f(y)|\,dy \leq$$

$$\leq \max_{y\in(a,b)} |f(y)|\,(b-a)$$



$$\sqrt{\ } \; |\psi(x)| < \frac{1}{\sqrt{2\pi}}$$

$$\times \max_{k'} |A_{k'}| \underbrace{(k_2 - k_1)}_{\Delta k}$$

$$A_k \approx \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-ikx}$$

$$\times \psi(x)\, dx$$

$$\sqrt{}\ |A_k| \lesssim \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} |\psi(x)|\, dx$$

$$|A_k| \lesssim \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sqrt{2\pi}}$$

$$\times \max_{k'} |A_{k'}|\ \Delta k\, (x_2 - x_1)$$

$$|A_k| \lesssim \frac{1}{2\pi} \max_{k'} |A_{k'}| \times \Delta k \, \Delta x$$

true for all k

choose k such that $|A_k| = \max_{k'} |A_{k'}|$

$$\cancel{|A_k|} \lesssim \frac{1}{2\pi} \cancel{|A_k|} \, \Delta x \, \Delta k$$

$$\frac{\Delta x \, \Delta k}{2\pi} \geq 1$$

$$\Delta x \, \Delta k \geq 2\pi \cancel{\times \hbar}$$

$$p = \frac{h}{\lambda} = \frac{h}{2\pi} \frac{2\pi}{\lambda}$$

$$= \hbar k$$

$$\Delta x \, \Delta p \geq \frac{h}{\cancel{2\pi}} \cancel{2\pi}$$

$$\Delta x \, \Delta p \geq h$$

Focus on t-dependen-ce

$$\psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_\omega \times e^{-i\omega t} \, d\omega$$

$$A_\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \times \psi(t) \, dt$$

$$\Delta\omega \Delta t \geq 2\pi$$

$$\times \hbar$$

$$\hbar\omega = \frac{h}{\cancel{2\pi}} \cancel{2\pi} \nu = h\nu$$

$$= E$$

$$\Delta E \Delta t \geq h$$

The Schrödinger equation

1. Linear

If ψ_1 and ψ_2 are solutions then $c_1\psi_1 + c_2\psi_2$ is also a solution.
(superposition)

2. Constants may depend on $\hbar, m, q$, etc., but not $E, p, k, \omega, \ldots$

3. Plane waves (e.g., $Ae^{i(kx-\omega t)}$) are solutions.

Linear PDEs

$$\boxed{\psi(x,t)}$$

Try wave equation (in 1D)!

$$\frac{\partial^2 \psi}{\partial x^2} = \gamma \frac{\partial^2 \psi}{\partial t^2}$$

It is a linear PDE.

ψ_1, ψ_2

$$\frac{\partial^2 \psi_i}{\partial x^2} = \gamma \frac{\partial^2 \psi_i}{\partial t^2}$$

$$i = 1, 2$$

$$\psi = c_1 \psi_1 + c_2 \psi_2$$

$$\frac{\partial^2 \psi}{\partial x^2} = \gamma \frac{\partial^2 \psi}{\partial t^2}$$

$$\psi(x,t) = A \times e^{i(kx-\omega t)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \underline{A e^{i(kx-\omega t)}} \times (ik)^2 = -k^2 \psi$$

$$\frac{\partial^2 \psi}{\partial t^2} = \underline{A e^{i(kx-\omega t)}} \times (-i\omega)^2 = -\omega^2 \psi$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2\psi$$

$$= \gamma \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \gamma \psi$$

$$k^2 = \gamma \omega^2$$

$$\gamma = \frac{k^2}{\omega^2} = \frac{\hbar^2 k^2}{\hbar^2 \omega^2}$$

$$= \frac{p^2}{E^2} = \frac{p^2}{(p^2/2m)^2}$$

$$\gamma = \frac{\cancel{p^2}}{p^{\cancel{4}\,2}}\, 4m^2$$

$$\gamma = \frac{4m^2}{p^2}$$

NOT GOOD

TRY

$$\frac{\partial^r \chi}{\partial x^r} = \gamma \frac{\partial^s \chi}{\partial t^s}$$

$$\psi(x,t) = A \times e^{i(kx-\omega t)}$$

$$\frac{\partial^r \psi}{\partial x^r} = \underline{A e^{i(kx-\omega t)}} \times (ik)^r$$

$$= (ik)^r \psi$$

$$\frac{\partial^s \psi}{\partial t^s} = (-i\omega)^s \psi$$

$$(ik)^r \cancel{\eta} = (-i\omega)^s g \times \cancel{\eta}$$

$$i^r k^r = g\,(-i)^s \omega^s$$

$$g = i^r k^r \underline{(-i)^{-s}}\, \omega^{-s}$$

$$\left(\frac{1}{i}\right)^{-s} = i^s$$

$$g = i^{r+s} k^{r} \omega^{-s}$$

$$g = i^{r+s} \frac{\hbar^{r} k^{r}}{\hbar^{s} \omega^{s}} \hbar^{s-r}$$

$$g = i^{r+s} \frac{p^{r}}{E^{s}} \hbar^{s-r}$$

$$E = \frac{p^{2}}{2m}$$

$$\gamma = i^{r+s} \, \frac{p^{r}}{\dfrac{p^{2s}}{2^{s} m^{s}}} \times h^{s-r}$$

$$\gamma = i^{r+s} \, h^{s-r} \, 2^{s} m^{s} \times p^{r-2s}$$

$$\gamma = 2^{s} i^{r+s} h^{s-r} \times m^{s} p^{r-2s} \quad \checkmark$$

$$r - 2s = 0 \Rightarrow r = 2s$$

$$s = 1 \Longrightarrow r = 2$$

$$\frac{\partial^2 \psi}{\partial x^2} = \gamma \frac{\partial \psi}{\partial t}$$

$$\gamma = 2 i^3 \hbar^{-1} m$$

$$\gamma = -2i \frac{m}{\hbar}$$

$$\boxed{\gamma = \frac{2m}{i\hbar}}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2m}{i\hbar}\frac{\partial \psi}{\partial t}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} \underset{\text{wave}}{\overset{\text{plane}}{=}} -\frac{\hbar^2}{2m}$$

$$\times (-k^2)\psi$$

$$= \frac{\hbar^2 k^2}{2m}\psi = \frac{p^2}{2m}\psi$$

$$i\hbar\frac{\partial \psi}{\partial t} \underset{\text{wave}}{\overset{\text{plane}}{=}} i\hbar$$

$$\times (-i\omega)\psi = E\psi$$

$$E\psi = \frac{p^2}{2m}\psi$$

3D:

$$\psi(\vec{r}, t) = A \times e^{i(\underbrace{\vec{k}\cdot\vec{r}}_{k_x x + k_y y + k_z z} - \omega t)}$$

$$\vec{p} = \hbar\vec{k}, \quad \vec{k} = \frac{2\pi}{\lambda}\hat{j}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi$$

$$\nabla^2 = \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Look at p. 129.

$$i\hbar \frac{\partial \psi}{\partial t} = \hbar \omega \psi$$

$$= E \psi$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = \frac{\hbar^2\overbrace{(k_x^2+k_y^2+k_z^2)}^{\vec{k}^2}}{2m}\psi$$

$$= \frac{\vec{p}^{\,2}}{2m}\psi$$

$$i\hbar\frac{\partial}{\partial t} \longrightarrow \text{"}\hat{E}\text{"}$$

$$-\frac{\hbar^2}{2m}\nabla^2 \longrightarrow \frac{\hat{\vec{p}}^2}{2m}$$

$$\hat{\vec{p}} = -i\hbar\vec{\nabla}$$

$$= \frac{\hbar}{i}\vec{\nabla}$$

$$\hat{P}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\hat{P}_l = \frac{\hbar}{i}\frac{\partial}{\partial q_l}$$

$$\vec{F} = -\vec{\nabla} V(\vec{r}, t)$$

$$"E" = \frac{\vec{p}^2}{2m} + V$$

$$E\psi = \frac{\vec{p}^2}{2m}\psi + V\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

$$+ \hat{V}\psi$$

$$\hat{V}\psi = V\psi$$

(multiplicative)

$$i\hbar \frac{\partial \psi}{\partial t} = \overbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right)}^{\text{Hamiltonian}}\psi$$

$$\psi = \psi(q_1, \dots, q_f, t)$$

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi}$$

$$\hat{H} = H\left(\hat{q}_1, \ldots, \hat{q}_f, \hat{p}_1, \ldots, \hat{p}_f, t\right)$$

$$\hat{p}_l = \frac{\hbar}{i}\frac{\partial}{\partial q_l}$$

$$\hat{q}_l = q_l \times$$

$$\hat{p}_l f = \frac{\hbar}{i} \frac{\partial f}{\partial q_l}$$

$$\hat{q}_l f = q_l f$$

N particles, no constraints

$$\hat{H} = \sum_{j=1}^{3N} \frac{\hat{p}_j^2}{2m_j} + \hat{V}(x_1, \ldots, x_{3N}, t)$$

$$\hat{H} = -\sum_{j=1}^{3N} \frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial x_j^2} + V(\hat{x}_1, \ldots, \hat{x}_{3N}, t)$$

$$= -\sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \vec{\nabla}_i^2 + \hat{V}$$

$$\boxed{\vec{\nabla}_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}\right)}$$

$$F = F(q_1, \ldots, q_f, p_1, \ldots, p_f, t)$$

$$\Big\downarrow$$

$$\hat{F} = F(\hat{q}_1, \ldots, \hat{q}_f, \hat{p}_1, \ldots, \hat{p}_f, t)$$

Operators must be linear

$$\hat{q}_l(c_1\gamma_1 + c_2\gamma_2)$$

$$= c_1 \hat{q}_l \gamma_1 + c_2 \hat{q}_l \gamma_2$$

$$\hat{p}_l(c_1\gamma_1 + c_2\gamma_2)$$

(p. 133)

$$= c_1 \hat{p}_l \gamma_1 + c_2 \hat{p}_l \gamma_2$$

$$\hat{F}[c_1\psi_1 + c_2\psi_2]$$

$$= c_1\hat{F}\psi_1 + c_2\hat{F}\psi_2$$

We will also require that all operators are

SELF-ADJOINT

Operators may not commute

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{A},\hat{B}]f = \hat{A}\hat{B}f - \hat{B}\hat{A}f = \hat{A}(\hat{B}f) - \hat{B}(\hat{A}f)$$

$$[\hat{q}_k, \hat{p}_\ell] :$$

$$[\hat{q}_k, \hat{p}_\ell] f = \hat{q}_k \hat{p}_\ell f$$

$$- \hat{p}_\ell \hat{q}_k f$$

$$= q_k \frac{\hbar}{i} \frac{\partial f}{\partial q_\ell}$$

$$- \frac{\hbar}{i} \frac{\partial}{\partial q_\ell} (q_k f)$$

$$= \cancel{\frac{\hbar}{i} q_k \frac{\partial f}{\partial q_l}}$$

$$- \frac{\hbar}{i} \frac{\partial q_k}{\partial q_l} f \qquad = \delta_{kl}$$

$$- \cancel{\frac{\hbar}{i} q_k \frac{\partial f}{\partial q_l}}$$

$$[\hat{q}_k, \hat{p}_l] f = i\hbar \delta_{kl} f$$

$$[\hat{q}_k, \hat{p}_l] = i\hbar \delta_{kl} \mathbb{1}$$

$$[\hat{q}_k, \hat{p}_l] = i\hbar\delta_{kl}$$

$$k \neq l \quad [\hat{q}_k, \hat{p}_l] = 0$$

$$[\hat{q}_l, \hat{p}_l] = i\hbar \, 1$$

CM $\longrightarrow$ QM

$$F \longrightarrow \hat{F}$$

$$\{F, G\} \to \frac{1}{i\hbar}[\hat{F}, \hat{G}]$$

$$\{q_k, p_l\} \overset{CM}{=} \delta_{kl}$$

$$[\hat{q}_k, \hat{p}_l] \overset{QM}{=} i\hbar\delta_{kl}$$

Dirac

$$\frac{1}{i\hbar}[\,,\,] \xrightarrow[\hbar \to 0]{} \{\,,\,\}$$

CM

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}$$

$$\frac{d\hat{F}}{dt} = \frac{\partial \hat{F}}{\partial t} + \frac{1}{i\hbar}[\hat{F}, \hat{H}]$$

QM (Heisenberg, Born, Jordan)