$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}$$

$$F = F(q, p, \underline{\underline{t}})$$

If $\frac{\partial F}{\partial t} = 0$ and $\{F, H\} = 0$

then $F = \text{const.}$

$$\text{CM} \longrightarrow \text{QM}$$

$$F \longrightarrow \hat{F}$$

$$\{F, G\} \longrightarrow \frac{1}{i\hbar}[\hat{F}, \hat{G}]$$

$$\hbar = \frac{h}{2\pi}$$

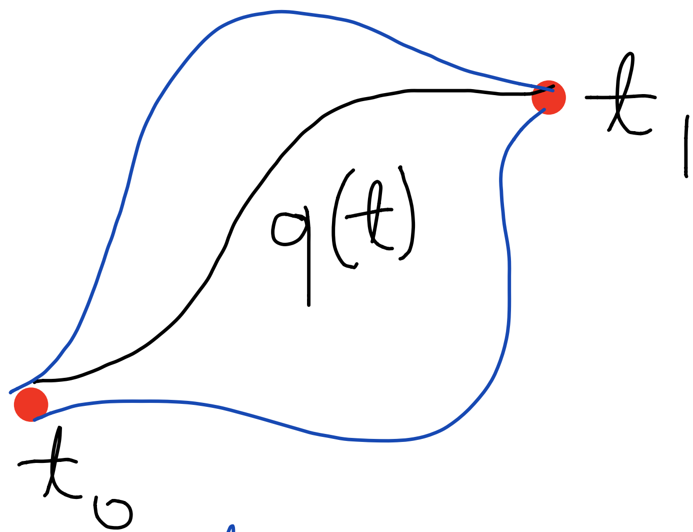
$$\frac{d\hat{F}}{dt} = \frac{\partial \hat{F}}{\partial t} + \frac{1}{i\hbar}[\hat{F}, \hat{H}]$$

Variational formulation of classical mechanics (Hamilton's principle, principle of least action)

$t_0 \longrightarrow t_1$

[deterministic]

teleological



$$S = \int_{t_0}^{t_1} \mathcal{L}\, dt$$

action integral

extremum of S

(minimum or maximum)

$\mathcal{L} = T - V$ or $T - U$

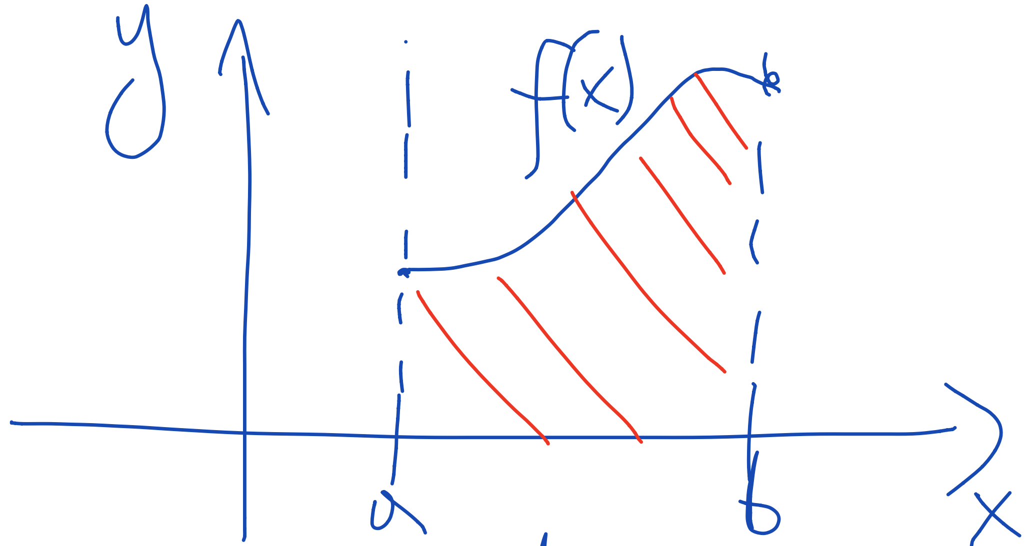
$$S = \int_{t_0}^{t_1} L(\overbrace{q_1, \dots, q_f}^{t}, \overbrace{\dot{q}_1, \dots, \dot{q}_f}^{t}, t)\,dt$$

$$S[q_1, \dots, q_f]$$

functional

$$f(x) : x \longrightarrow f(x)$$

$$F[f] : f \longrightarrow F[f]$$



$$\text{area} = \int_a^b f(x)\,dx$$

$$f(x_1, \dots, x_n)$$

$$\left.\frac{\partial f}{\partial x_i}\right|_{x=x_0} = 0$$

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,dx_i$$

$$df = 0$$

$$f(x) - f(x_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_{x_0} (x_i - x_{i0})$$

$x = (x_1, \ldots, x_n)$

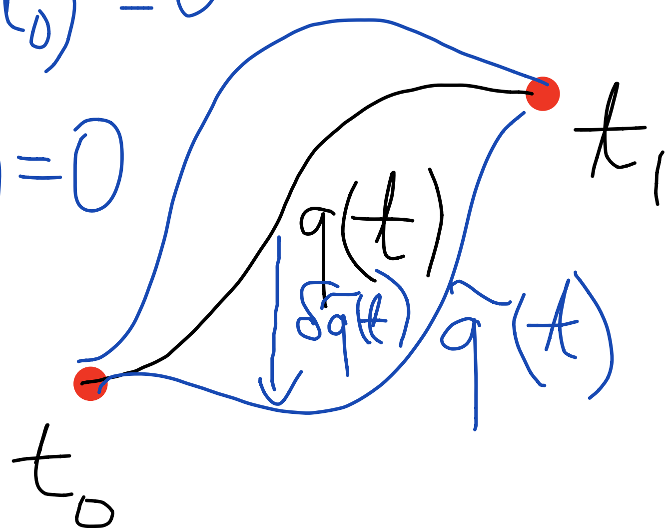
$$+ \ldots$$

0

Linear in displacements

$\delta q_l(t_0) = 0$

$\delta_l(t_1) = 0$



$$\tilde{q}_l(t) = q_l(t) + \delta q_l(t)$$

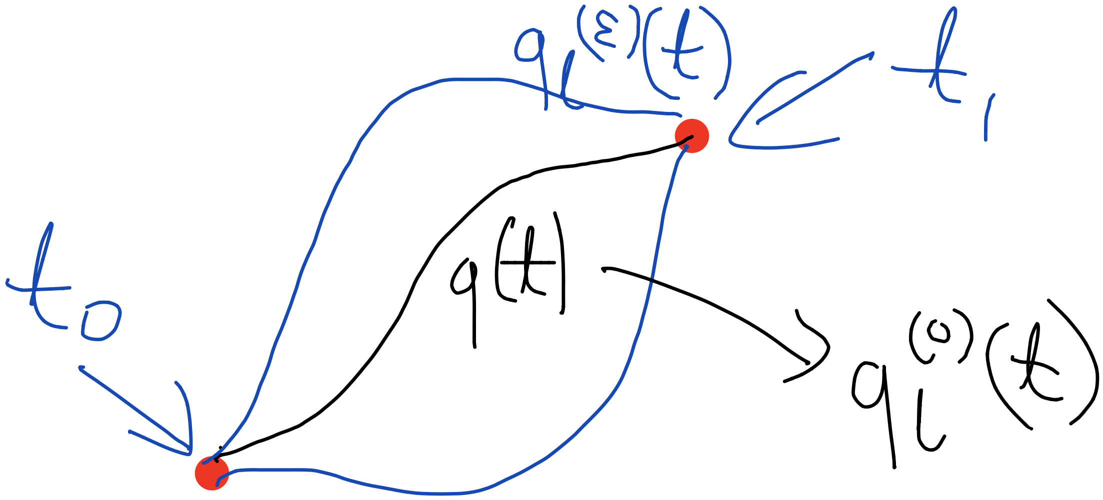
$$S[\tilde{q}] - S[q] = \underline{\delta S[q]} + \dots$$

linear in δq

1st variation

$$\boxed{\delta S[q] = 0}$$

$$\delta q_l(t_0) = \delta q_l(t_1) = 0, \quad l = 1, \dots, f$$



$$q_l^{(\varepsilon)}(t) \equiv q_l(t, \varepsilon)$$

$\varepsilon = 0 \Rightarrow$ physical

$$q_l^{(0)}(t) \equiv q_l(t, 0)$$
$$= q_l(t)$$

$$q_l^{(\varepsilon)}(t_0) \equiv q_l(t_0, \varepsilon) = q_l(t_0)$$

$$q_l^{(\varepsilon)}(t_1) \equiv q_l(t_1, \varepsilon) = q_l(t_1)$$

$$S[q^{(\varepsilon)}] \equiv S(\varepsilon)$$

$$= \int_{t_0}^{t_1} L\left(q^{(\varepsilon)}(t), \dot{q}^{(\varepsilon)}(t), t\right) dt$$

$$\delta S[q] = 0$$

$$\delta q_\ell = q_\ell^{(\varepsilon)}(t) - q_\ell(t)$$

$$= q_\ell(t, \varepsilon) - q_\ell(t, 0)$$

$$= \left.\frac{\partial q_\ell(t, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon + \dots$$

$$S[q^{(\varepsilon)}] - S[q^{(0)}]$$

$$= \left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + \dots$$

$$\left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

$$\frac{dS}{d\varepsilon} = \frac{d}{d\varepsilon} \int_{t_0}^{t_1} L(q(t,\varepsilon), \dot{q}(t,\varepsilon), t)\, dt$$

$$= \int_{t_0}^{t_1} \frac{\partial L}{\partial \varepsilon}\, dt$$

$$\frac{dS}{d\varepsilon} = \int_{t_0}^{t_1} \sum_{l=1}^{f} \left(\frac{\partial L}{\partial q_l^{(\varepsilon)}} \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon} + \frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}} \frac{\partial \dot{q}_l^{(\varepsilon)}}{\partial \varepsilon} \right) dt$$

$$\frac{dS}{dt} = \sum_{l=1}^{f} \left\{ \int_{t_0}^{t_1} \frac{\partial L}{\partial q_l^{(\varepsilon)}} \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon} dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}} \underbrace{\frac{\partial \dot{q}_l^{(\varepsilon)}}{\partial \varepsilon}} dt \right\} \checkmark$$

$$\frac{\partial \dot{q}_l^{(\varepsilon)}}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon} \frac{d}{dt} q_l^{(\varepsilon)} = \frac{d}{dt} \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}$$

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}} \frac{\partial \dot{q}_l^{(\varepsilon)}}{\partial \varepsilon} dt$$

$$= \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}} \frac{d}{dt}\left(\frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}\right) dt$$

$$\underset{\text{parts}}{\overset{\text{by}}{=}} \left. \frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}} \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon} \right|_{t_0}^{t_1}$$

$$- \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}}\right) \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon} dt$$

$$q_l^{(\varepsilon)}(t_0) = q_l(t_0, \varepsilon)$$
$$= q_l(t_0)$$

$$\left.\frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}\right|_{t=t_0} = 0$$

$$q_l^{(\varepsilon)}(t_1) = q_l(t_1, \varepsilon)$$
$$= q_l(t_1)$$

$$\left.\frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}\right|_{t=t_1} = 0$$

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}} \frac{\partial \dot{q}_l^{(\varepsilon)}}{\partial \varepsilon} dt =$$

$$= - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}}\right) \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}\, dt$$

$$\frac{dS}{d\varepsilon} = \sum_{l=1}^{f} \left\{ \int_{t_0}^{t_1} \frac{\partial L}{\partial q_l^{(\varepsilon)}} \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}\, dt \right.$$

$$\left. - \int_{t_0}^{t_1} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}} \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}\, dt \right\}$$

$$= \sum_{l=1}^{f} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_l^{(\varepsilon)}} \checkmark - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l^{(\varepsilon)}} \checkmark \right) \times \frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon} \checkmark\, dt$$

$$\frac{dS}{d\varepsilon}\bigg|_{\varepsilon=0} = 0$$

$$\frac{dS}{d\varepsilon}\bigg|_{\varepsilon=0} = \sum_{l=1}^{f} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_l} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} \right) \underline{\frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}\bigg|_{\varepsilon=0}} \, dt$$

$$\frac{\partial q_l^{(\varepsilon)}(t)}{\partial \varepsilon}\bigg|_{\varepsilon=0}$$ can be made arbitrary

V $q_l^{(\varepsilon)}(t) = q_l(t) + \varepsilon \eta_l(t)$

arbitrary

$$\eta_l(t_0) = \eta_l(t_1) = 0$$

$$q_l^{(\varepsilon)}(t_0) = q_l(t_0)$$

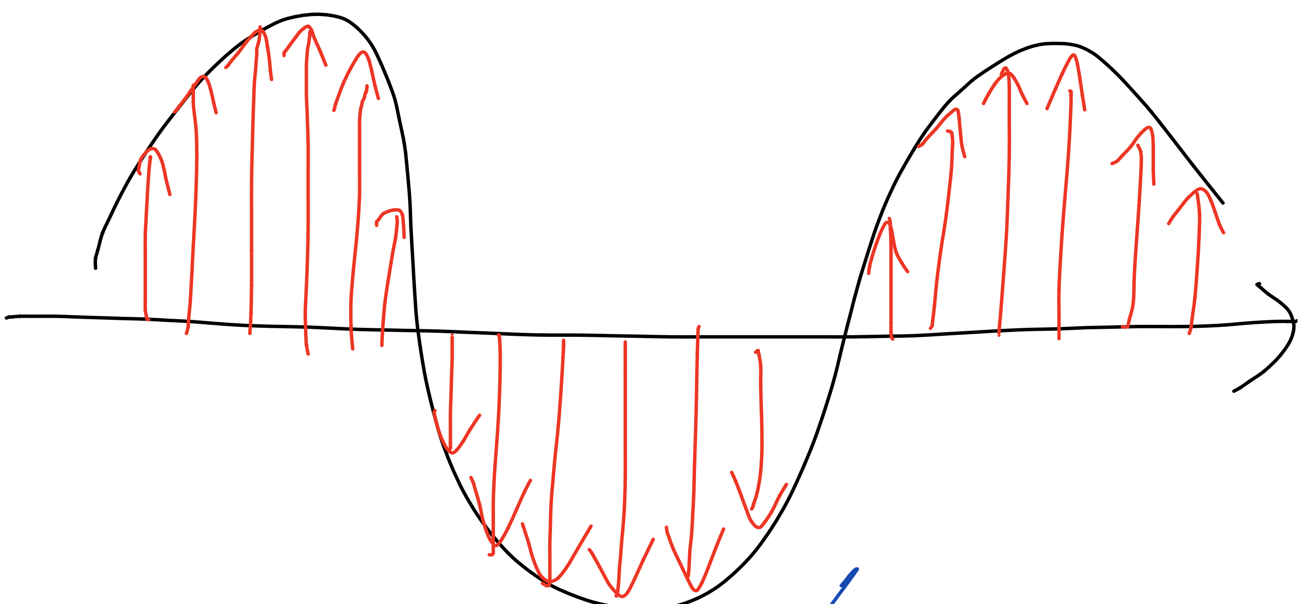
$$q_l^{(\varepsilon)}(t_1) = q_l(t_1)$$

$$q_l^{(0)}(t) = q_l(t)$$

$$\left.\frac{\partial q_l^{(\varepsilon)}}{\partial \varepsilon}\right|_{\varepsilon=0} = \eta_l(t)$$

$$\frac{\partial L}{\partial q_l} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} = 0$$

$l = 1, \dots, f$



$$S = \int_{t_0}^{t_1} L \, dt$$

$\boxed{\text{Black-Body radiation}}$

✓ $$u_\nu(T) = \frac{8\pi\nu^2}{c^3} kT$$

density of energy at ν

$\boxed{\text{Rayleigh-Jeans law}}$

$u_\nu(T)\,d\nu \rightarrow$ density of energy in $[\nu, \nu + d\nu]$

$$\frac{\lambda}{T} = \lambda\nu = c \qquad \nu = \frac{1}{T}$$

$$\nu = \frac{c}{\lambda}$$

$$u_\nu(T)\,\underline{d\nu} = \rho_\lambda(T)\,\underline{d\lambda}$$

$$d\nu = c\,d\left(\frac{1}{\lambda}\right) = \left|-\frac{c}{\lambda^2}\right| d\lambda$$

$$= \frac{c}{\lambda^2}\,d\lambda$$

$$u_\nu(T)\,d\nu = \frac{8\pi\nu^2}{c^3}kT\,d\nu$$

$$= \frac{8\pi\,\cancel{c^2}}{\lambda^2\,\cancel{c^3}}kT\,\frac{\cancel{c}}{\lambda^2}\,d\lambda$$

$$= \left(\frac{8\pi}{\lambda^4} kT \right) d\lambda$$

$$\boxed{\rho_\lambda(T) = \frac{8\pi}{\lambda^4} kT}$$

$$\int_0^\infty u_\nu(T)\, d\nu = \infty$$

total density of energy

(ultraviolet catastrophe)

$$E_n = nh\nu \quad , \quad n = 0, 1, \ldots$$

Planck constant

$$u_\nu(T) = \frac{8\pi h\nu^3}{c^3} \times$$

$$\times \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$\nu \to 0 \quad (\lambda \to \infty)$$

$$u_\nu(T) = \frac{8\pi h\nu^3}{c^3}$$

$$\times \frac{1}{\cancel{1} + \frac{h\nu}{kT} + \ldots - \cancel{1}}$$

$$u_\nu(T) \approx \frac{8\pi \cancel{h} \nu^{\cancel{3}\,2}}{c^3} \frac{kT}{\cancel{h\nu}}$$

$$= \frac{8\pi\nu^2}{c^3} kT$$

$$\lim_{\nu \to \infty} u_\nu(T) = \infty$$

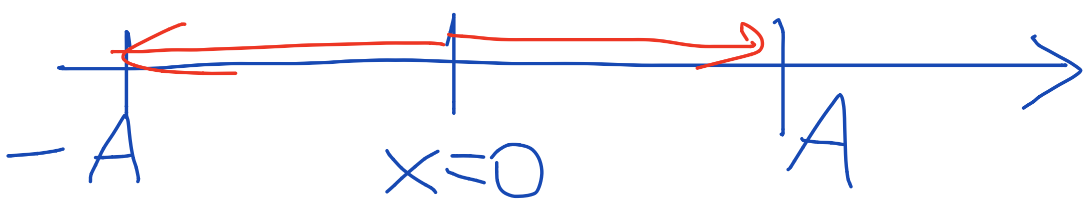
$$\boxed{\text{December 14th 1900}}$$

1906

$$\rightarrow J_\ell = \oint p_\ell \, dq_\ell$$

action variables

$$J = 2\int_{-A}^{A} p_x \, dx$$



If $J = nh$ then

$$E = nh\nu$$

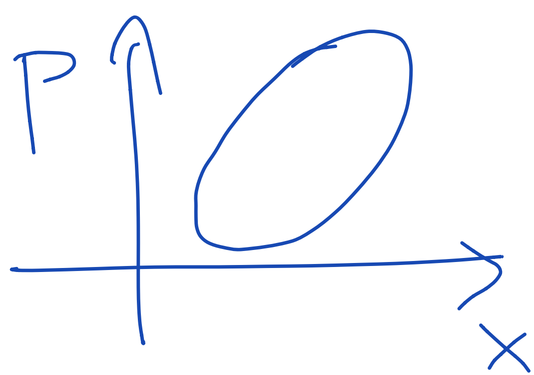
$$x(t) = A \sin \omega t$$

$$\omega = \frac{2\pi}{T} = 2\pi\nu$$

$p_x = m\dot{x}$, etc. (see p. 79)

$$J = \oint p\,dx$$

$$= 2\int_{-A}^{A} p\,dx$$



$$E = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$\omega = 2\pi\nu$$

$$= \frac{2\pi}{T}$$

$$p = \sqrt{2m\left(E - \frac{m\omega^2 x^2}{2}\right)}$$

$$= \left(2mE - m^2\omega^2 x^2\right)^{\frac{1}{2}}$$

$$J = 2\int_{-A}^{A} \left(2mE - m^2\omega^2 x^2\right)^{\frac{1}{2}} dx$$

$$= 2\sqrt{2mE}\int_{-A}^{A} \left(1 - \underbrace{\frac{m^2\omega^2}{2mE}}_{c^2} x^2\right)^{\frac{1}{2}} dx$$

$$c = \frac{m\omega}{\sqrt{2mE}}$$

$$J = 2\sqrt{2mE}\int_{-A}^{A} \left(1 - (cx)^2\right)^{\frac{1}{2}} dx$$

$$y = cx$$

$$dx = \frac{1}{c}\,dy$$

$$|y| = |cx| \leq \frac{m\omega}{\sqrt{2mE}} A$$

$$= \sqrt{\frac{\cancel{m}^2 \omega^2 A^2}{2\cancel{m}E}} =$$

$$= \sqrt{\frac{m\omega^2 A^2}{2E}} = 1$$

$$E = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2$$

$x = A, \quad p = 0$

$$E = \frac{m\omega^2}{2} A^2$$

$$J = 2\sqrt{2mE}\,\frac{1}{c} \times \int_{-1}^{1} (1-y^2)^{\frac{1}{2}}\, dy$$

$y = \sin\alpha, \quad dy = \cos\alpha\, d\alpha$

$\frac{\pi}{2}$

$$J = 2\sqrt{2mE}\,\frac{1}{c} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\alpha\, d\alpha$$

$$J = \cancel{2}\sqrt{2mE}\,\frac{1}{c}\cdot\frac{\pi}{\cancel{2}}$$

$$= \frac{\left(\sqrt{2mE}\right)^2}{m\omega}\cdot\pi$$

$$c = \frac{m\omega}{\sqrt{2mE}}$$

$$J = \frac{2\cancel{m}E}{\cancel{m}\omega}\,\pi$$

$$= \frac{2E}{\omega}\,\pi$$

$$\omega = 2\pi\nu$$

$$J = \frac{\cancel{2}E}{\cancel{2}\cancel{\pi}\nu}\cancel{\pi} = \frac{E}{\nu}$$

$$\boxed{E = J\nu}$$

Planck, 1906

$$J = nh \Rightarrow E = n\underline{\underline{h\nu}}$$