$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} = 0$$

$$l = 1, \ldots, f$$

$$L = T - V \text{ or } T - U$$

$$L = L(q, \dot{q}, t)$$

$$= T(\dot{x}(q, \dot{q}, t)) - V(x(q, t), t)$$

$$[\text{or } T - U(x(q,t), \dot{x}(q, \dot{q}, t), t)]$$

f 2nd-order ODEs

$$q_l(t_0) = q_{l0}, \; \dot{q}_l(t_0) = v_{l0}$$

$$l = 1, \ldots, f$$

Example: $q_1 = r,\ q_2 = \vartheta,$
$q_3 = \varphi$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \sin^2\vartheta \, \dot{\varphi}^2 \right)$$

$$\begin{cases} v_l = \dot{q}_l \\ \dfrac{d}{dt} \dfrac{\partial L}{\partial v_l} - \dfrac{\partial L}{\partial q_l} = 0 \end{cases} \qquad l = 1, \dots, f$$

conjugate (generalized) momenta

$$p_l = \frac{\partial L}{\partial \dot{q}_l}, \quad l = 1, \dots, f$$

$$\boxed{\dot{p}_l = \frac{\partial L}{\partial q_l}}$$

Assume $L = T - V$

$$\frac{\partial V}{\partial \dot{q}_l} = 0$$

$$p_l = \frac{\partial L}{\partial \dot{q}_l} = \frac{\partial T}{\partial \dot{q}_l}$$

$$= \sum_{j=1}^{3N} \frac{\partial T}{\partial \dot{x}_j} \frac{\partial \dot{x}_j}{\partial \dot{q}_l} \quad \checkmark$$

$$T = \sum_{j=1}^{3N} \frac{m_j \dot{x}_j^2}{2}$$

$$\frac{\partial T}{\partial \dot{x}_j} = \frac{m_j}{\cancel{2}} \cancel{2} \dot{x}_j$$

$$= m_j \dot{x}_j$$

$$p_l = \sum_{j=1}^{3N} \underbrace{m_j \dot{x}_j}_{p_{x_j}} \frac{\partial x_j}{\partial q_l}$$

$$p_l = \sum_{j=1}^{3N} p_{x_j} \frac{\partial x_j}{\partial q_l}$$

$$Q_l = \sum_{j=1}^{3N} X_j \frac{\partial x_j}{\partial q_l}$$

N particles, no constraints

$$q_j = x_j, \; j=1,\dots,3N$$

$$p_j = p_{x_j} = m_j \dot{x}_j$$

Let's assume now that

$$L = T - U$$

$$p_l = \frac{\partial L}{\partial \dot{q}_l} = \frac{\partial T}{\partial \dot{q}_l} - \frac{\partial U}{\partial \dot{q}_l}$$

$$p_j = \underbrace{\frac{\partial T}{\partial \dot{x}_j}}_{m_j \dot{x}_j} - \frac{\partial U}{\partial \dot{x}_j}$$

$$\boxed{p_j = m_j \dot{x}_j - \frac{\partial U}{\partial \dot{x}_j}}$$

Example:

$$U = q\varphi - \frac{q}{c} \underbrace{\vec{A} \cdot \vec{v}}_{A_x \dot{x} + A_y \dot{y} + A_z \dot{z}}$$

$$p_x = m\dot{x} + \frac{q}{c}A_x$$

$$p_y = m\dot{y} + \frac{q}{c}A_y$$

$$p_z = m\dot{z} + \frac{q}{c}A_z$$

$$\boxed{\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}}$$

Lagrange $\longrightarrow$ Hamilton

$q, v \longrightarrow \underbrace{q, p}_{\text{phase space}}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} = 0$$

$l = 1, \ldots, f$

2nd order ODEs

$\longrightarrow$ 1st order ODEs

$$\begin{cases} v_l = \dot{q}_l \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial v_l} - \dfrac{\partial L}{\partial q_l} = 0 \end{cases}$$

$$(q_l, v_l) \longrightarrow (q_l, p_l)$$

$$L \longrightarrow H$$

$$p_l = \frac{\partial L(q, \boxed{v}, t)}{\partial v_l} \qquad l = 1, \dots, f$$

$$v_l = v_l(q, p, t)$$

$$\begin{cases} f_1(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \\ \dots \\ f_n(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \end{cases}$$

$$x_i = x_i(y_1, \dots, y_m) \qquad i = 1, \dots, n$$

$$\det \left\| \frac{\partial f_i}{\partial x_j} \right\|_{i,j=1,\dots,n} \neq 0$$

$$\det \left\| \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_l} \right\| \neq 0$$

Example:

$$p_r = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\vartheta = m r^2 \dot{\vartheta}$$

$$\dot{\vartheta} = \frac{p_\vartheta}{m r^2}$$

$$p_\varphi = m r^2 \sin^2\vartheta \, \dot{\varphi}$$

$$\dot{\varphi} = \frac{p_\varphi}{m r^2 \sin^2\vartheta}$$

$$p_x = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

$$p_j = m_j \dot{x}_j \Rightarrow \dot{x}_j = \frac{p_j}{m_j}$$

$$v_l = v_l(q, p, t)$$

new new old

$$\boxed{H(q,p,t) = \sum_{l=1}^{f} p_l v_l(q,p,t) - L(q, v(q,p,t), t)}$$

old

Legendre's transformation

$$H = \sum_{k=1}^{f} p_k v_k(q,p,t) - L(q, v(q,p,t), t)$$

$$\frac{\partial H}{\partial p_l} = v_l + \cancel{\sum_k p_k \frac{\partial v_k}{\partial p_l}} - \cancel{\sum_k \underbrace{\frac{\partial L}{\partial v_k}}_{p_k} \frac{\partial v_k}{\partial p_l}}$$

$$= v_l$$

$$\frac{\partial H}{\partial q_l} = \cancel{\sum_k p_k \frac{\partial v_k}{\partial q_l}}$$

$$- \frac{\partial L}{\partial q_l} - \cancel{\sum_k \underbrace{\frac{\partial L}{\partial v_k}}_{p_k} \frac{\partial v_k}{\partial q_l}}$$

$$= - \frac{\partial L}{\partial q_l}$$

$$\checkmark \quad \frac{\partial H}{\partial p_l} = v_l \, , \quad \frac{\partial H}{\partial q_l} = - \frac{\partial L}{\partial q_l}$$

$$\frac{d}{dt}\frac{\partial L}{\partial q_l} - \frac{\partial L}{\partial q_l} = 0$$

$$\dot{p}_l = \frac{\partial L}{\partial q_l}$$

$$\dot{p}_l = -\frac{\partial H}{\partial q_l}$$

$$\dot{q}_l = v_l = \frac{\partial H}{\partial p_l}$$

$$\dot{q}_l = \frac{\partial H}{\partial p_l}$$

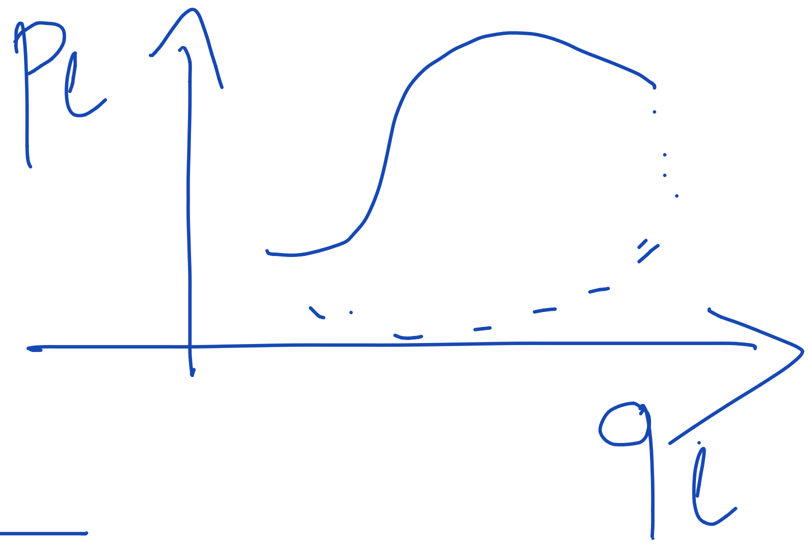
$$\dot{p}_l = -\frac{\partial H}{\partial q_l}$$

$$l = 1, \ldots, f$$

- 1st order ODEs
- initial conditions

$$\begin{cases} q_l(t_0) = q_{l0} \\ p_l(t_0) = p_{l0} \end{cases} \quad l = 1, \ldots, f$$

$$q_1(t), \ldots, q_f(t), p_1(t), \ldots, p_f(t)$$



Theorem: If constraints do not depend explicitly on t and forces are derived from V, then

$$H = T + V$$

$$H(q,p,t) = T(\dot{x}(q,p)) + V(x(q),t)$$

Proof:

$$T = \frac{1}{2}\sum_{j=1}^{3N} m_j \dot{x}_j^2$$

$$x_j = x_j(q_1, \ldots, q_f, \cancel{t})$$

$$\dot{x}_j = \sum_k \frac{\partial x_j}{\partial q_k}\dot{q}_k$$

$$T = \frac{1}{2}\sum_{j=1}^{3N} m_j \left(\sum_{k=1}^{f} \frac{\partial x_j}{\partial q_k}\dot{q}_k\right)^2$$

$$= \frac{1}{2}\sum_{j=1}^{3N} m_j \sum_{k,l=1}^{f} \frac{\partial x_j}{\partial q_k}\frac{\partial x_j}{\partial q_l} \times \dot{q}_k\dot{q}_l$$

$$T = \sum_{k,l=1}^{f} a_{kl}(q)\, \dot{q}_k \dot{q}_l$$

$$a_{kl}(q) = \frac{1}{2} \sum_{j=1}^{3N} m_j \frac{\partial x_j}{\partial q_k} \frac{\partial x_j}{\partial q_l}$$

T is a bilinear function of $\dot{q}_l$s (or v_ls)

$$T = \sum_{k,l=1}^{f} a_{kl}(q)\, v_k v_l$$

T is a homogeneous function of order 2

$f(x_1, \ldots, x_n)$ is homogeneous of order α if

$$f(sx_1, \ldots, sx_n) = s^\alpha \times f(x_1, \ldots, x_n)$$

$$T(sv_1, \ldots, sv_f)$$
$$= \sum_{k,l} a_{kl}(q)(sv_k)(sv_l)$$
$$= s^2 \sum_{k,l} a_{kl}(q) v_k v_l$$
$$= s^2 T$$

Euler's theorem:

function f is homogeneous of order α iff

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i = \alpha f$$

Proof: $\Longrightarrow$ $\quad y_i = s x_i$

$$\frac{d f(\overbrace{s x_1}^{y_1}, \ldots, \overbrace{s x_n}^{y_n})}{ds}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\bigg|_{y_i = s x_i} \underbrace{x_i}_{\frac{dy_i}{ds}}$$

$$f(sx_1, \ldots, sx_n) = s^\alpha$$
$$\times f(x_1, \ldots, x_n)$$

$$\frac{d f(sx_1, \ldots, sx_n)}{ds} = \alpha s^{\alpha - 1}$$
$$\times f(x_1, \ldots, x_n)$$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\bigg|_{y_i = sx_i} x_i$$

$$= \alpha s^{\alpha-1} f(x_1, \ldots, x_n)$$

$s = 1$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i = \alpha f(x_1, \ldots, x_n)$$

For T, $\alpha = 2$

$$\sum_{l=1}^{f} \frac{\partial T}{\partial v_l} v_l = 2T$$

$T - V$

$$H = \sum_l p_l v_l - L$$

$$= \sum_l p_l v_l - T + V$$

$$p_l = \frac{\partial L}{\partial v_l} = \frac{\partial T}{\partial v_l}$$

$$= \underbrace{\sum_l \frac{\partial T}{\partial v_l} v_l}_{2T} - T + V$$

$$= 2T - T + V$$

$$= T + V \quad \therefore$$

Example:

N particles, no constraints,

$q_j = x_j, \; j = 1, \ldots, 3N$

forces from V

$$H = T + V$$

$$T = \frac{1}{2} \sum_{j=1}^{3N} m_j \dot{x}_j^2$$

$$\dot{x}_j = \frac{p_j}{m_j}$$

$$T = \sum_{j=1}^{3N} \frac{p_j^2}{2m_j}$$

$$\boxed{H = \sum_{j=1}^{3N} \frac{p_j^2}{2m_j} + V(x,t)}$$

What about EM field?

$$\cancel{H = T + U}$$

$$H = \sum_{\ell} p_\ell \dot{q}_\ell - L$$

$$p_x = m\dot{x} + \frac{q}{c}A_x$$

$$p_y = m\dot{y} + \frac{q}{c}A_y$$

$$p_z = m\dot{z} + \frac{q}{c}A_z$$

$$\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}$$

$$H = \overbrace{p_x v_x + p_y v_y + p_z v_z}^{\vec{p}\cdot\vec{v}}$$

$$-T + U$$

$$(L = T - U)$$

$$U = q\varphi - \frac{q}{c}\vec{A}\cdot\vec{v} \quad \checkmark$$

$$\vec{v} = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right) \checkmark$$

$$H = \vec{p}\cdot\vec{v} - L$$

$$= \frac{1}{m}\vec{p}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)$$

$$- \frac{m}{2}\left[\frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)\right]^2 \checkmark$$

$$+ q\varphi - \frac{q}{mc}\vec{A}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)$$

$$= \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)$$

$$- \frac{\cancel{m}}{2m^{\cancel{2}}}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\varphi$$

$$H = \frac{1}{2m}\underbrace{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}_{} + q\varphi$$

$$q_i(t),\ p_i(t)\ \checkmark$$

$$F = F(q, p, t)$$

observable

(dynamical variable)

$$F(t) = F(q(t), p(t), t)$$

$$\frac{dF}{dt} = \sum_{l=1}^{f}\left(\frac{\partial F}{\partial q_l}\dot{q}_l + \frac{\partial F}{\partial p_l}\dot{p}_l\right) + \frac{\partial F}{\partial t}$$

$$= \sum_{l=1}^{f}\left(\frac{\partial F}{\partial q_l}\frac{\partial H}{\partial p_l} - \frac{\partial F}{\partial p_l}\frac{\partial H}{\partial q_l}\right) + \frac{\partial F}{\partial t} \quad \checkmark$$

$$\checkmark \quad \{F, G\} = \sum_{l=1}^{f}\left(\frac{\partial F}{\partial q_l}\frac{\partial G}{\partial p_l} - \frac{\partial F}{\partial p_l}\frac{\partial G}{\partial q_l}\right)$$

Poisson bracket

$$\boxed{\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}}$$

$$F = q_l$$

$$\dot{q}_l = \frac{dq_l}{dt} = \{q_l, H\} + \cancelto{0}{\frac{\partial q_l}{\partial t}}$$

$$\{q_l, H\} = \sum_k \left(\underbrace{\frac{\partial q_l}{\partial q_k}}_{\delta_{kl}} \frac{\partial H}{\partial p_k} - \cancelto{0}{\frac{\partial q_l}{\partial p_k}} \frac{\partial H}{\partial q_k} \right)$$

$$= \frac{\partial H}{\partial p_l}$$

$$\dot{q}_\ell = \frac{\partial H}{\partial p_\ell}$$

$$F = p_\ell$$

$$\dot{p}_\ell = \{p_\ell, H\} + \cancelto{0}{\frac{\partial p_\ell}{\partial t}}$$

$$\{p_\ell, H\} = \sum_k \left(\cancelto{0}{\frac{\partial p_\ell}{\partial q_k}} \frac{\partial H}{\partial p_k} - \underbrace{\frac{\partial p_\ell}{\partial p_k}}_{\delta_{k\ell}} \frac{\partial H}{\partial q_k} \right) = -\frac{\partial H}{\partial q_\ell}$$

$$\dot{p}_l = -\frac{\partial H}{\partial q_l}$$

$$\frac{dF}{dt} = \{F,H\} + \frac{\partial F}{\partial t}$$

$F = H$

$$\frac{dH}{dt} = \{H,H\} + \frac{\partial H}{\partial t}$$

$$\{F,F\} = 0$$

$$\left(\{F,G\} = -\{G,F\}\right)$$

etc.

$$\dot{H} = \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$$\dot{q}_l = \frac{\partial H}{\partial p_l}$$

$$\dot{p}_l = - \frac{\partial H}{\partial q_l}$$

t & H are conjugate

If $\frac{\partial H}{\partial t} = 0$ (no explicit t-dependence)

$\Rightarrow \dot{H} = 0 \Rightarrow H = \text{const.}$

Assume that we have conservative forces and constraints do not depend on time:

$$H = T + V$$

$$T(q, \dot{q}) \qquad V(q)$$

$$\Rightarrow \quad \frac{\partial H}{\partial t} = 0 \Rightarrow H = \text{const.}$$

$$\Rightarrow T + V = \text{const.}$$

If $\{F, H\} = 0$

and $\dfrac{\partial F}{\partial t} = 0 \implies$

$\implies \dfrac{dF}{dt} = 0 \implies F = \text{const.}$