$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$[x, p] = i\hbar$$

$$Q = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x \quad \checkmark$$

$$P = \frac{p}{(\hbar \omega m)^{\frac{1}{2}}} \quad \checkmark$$

$$H = \hbar\omega \mathcal{H}$$

$$\mathcal{H} = \frac{1}{2}(P^2 + Q^2)$$

$$[Q, P] = i$$

$$a = \frac{1}{\sqrt{2}}(Q + iP)$$ annihilation operator

$$a^{+} = \frac{1}{\sqrt{2}}(Q - iP)$$ creation operator

$$H = \hbar\omega\mathcal{H}$$

$$\mathcal{H} = N + \frac{1}{2}$$

$$N = a^{\dagger}a$$

number operator

$$[a, a^{\dagger}] = 1$$

$$H|n\rangle = E_n|n\rangle$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$|n\rangle = (n!)^{-\frac{1}{2}}(a^{\dagger})^n|0\rangle \quad \checkmark$$

$$n = 0, 1, 2, \ldots$$

$$N|n\rangle = n|n\rangle$$

$$a|n\rangle = n^{\frac{1}{2}}|n-1\rangle, \quad a^{\dagger}|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$$

How do the ket states $|n\rangle$ look like in a coordinate representation?

$$\langle Q|0\rangle = u_0(Q)$$

$$a|0\rangle = 0 \quad \checkmark\checkmark$$

$$Q = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x$$

$$P = \frac{p}{(\hbar\omega m)^{\frac{1}{2}}} = \frac{\hbar}{i} \frac{1}{(\hbar\omega m)^{\frac{1}{2}}} \frac{d}{dx}$$

$$\frac{df}{dx} = \frac{df}{dQ}\frac{dQ}{dx} = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} \frac{df}{dQ}$$

$$\frac{d}{dx} = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} \frac{d}{dQ}$$

$$P = \frac{\cancel{\hbar}}{i}\frac{1}{(\cancel{\hbar}\cancel{\omega}\cancel{m})^{\frac{1}{2}}}\left(\frac{\cancel{m}\cancel{\omega}}{\cancel{\hbar}}\right)^{\frac{1}{2}}\frac{d}{dQ}$$

$$= \frac{1}{i}\frac{d}{dQ} = -i\frac{d}{dQ}$$

$$a = \frac{1}{\sqrt{2}}(Q + iP)$$

$$= \frac{1}{\sqrt{2}}\left[Q + i(-i)\frac{d}{dQ}\right]$$

$$= \frac{1}{\sqrt{2}}\left(Q + \frac{d}{dQ}\right)$$

$$a^{+} = \frac{1}{\sqrt{2}}(Q - iP)$$

$$= \frac{1}{\sqrt{2}}\left[Q - i(-i)\frac{d}{dQ}\right]$$

$$= \frac{1}{\sqrt{2}}\left(Q - \frac{d}{dQ}\right)$$

$$a = \frac{1}{\sqrt{2}}\left(Q + \frac{d}{dQ}\right)$$

$$a^{+} = \frac{1}{\sqrt{2}}\left(Q - \frac{d}{dQ}\right)$$

$$\left(\frac{d}{dQ}\right)^{+} = -\frac{d}{dQ}$$

$$a|0\rangle = 0$$

$$a\, u_0(Q) = 0$$

$$\left(Q + \frac{d}{dQ}\right) u_0(Q) = 0$$

$$u_0(Q) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}Q^2}$$

$$\int_{-\infty}^{\infty} |u_0(Q)|^2 dQ = 1$$

$$|n\rangle = (n!)^{-\frac{1}{2}}(a^{+})^{n}|0\rangle$$

$$u_n(Q) = \langle Q|n\rangle$$

$$= (n!)^{-\frac{1}{2}}\left(\frac{1}{\sqrt{2}}\right)^{n}\left(Q-\frac{d}{dQ}\right)^{n}u_0(Q)$$

$$u_n(Q) = (n!)^{-\frac{1}{2}}\,2^{-\frac{n}{2}}\,\pi^{-\frac{1}{4}}$$

$$\times\left(Q-\frac{d}{dQ}\right)^{n} e^{-\frac{1}{2}Q^2}$$

$$Q-\frac{d}{dQ} = -e^{\frac{1}{2}Q^2}\frac{d}{dQ}e^{-\frac{1}{2}Q^2}$$

$$u_n(Q) = \left(2^n n! \, \pi^{\frac{1}{2}}\right)^{-\frac{1}{2}}$$

$$\left.\begin{aligned}
&\times (-1)^n \left(e^{\frac{1}{2}Q^2} \frac{d}{dQ} \cancel{e^{-\frac{1}{2}Q^2}}\right) \\
&\times \left(\cancel{e^{\frac{1}{2}Q^2}} \frac{d}{dQ} \cancel{e^{-\frac{1}{2}Q^2}}\right) \times \cdots \\
&\times \left(\cancel{e^{\frac{1}{2}Q^2}} \frac{d}{dQ} e^{-\frac{1}{2}Q^2}\right)
\end{aligned}\right\} n \text{ times}$$

$$\times \, e^{-\frac{1}{2}Q^2}$$

$$u_n(Q) = \left(2^n n! \, \pi^{\frac{1}{2}}\right)^{-\frac{1}{2}} (-1)^n$$

$$\times \, e^{\frac{1}{2}Q^2} \frac{d^n}{dQ^n} e^{-Q^2}$$

$$\downarrow$$

$$e^{-\frac{1}{2}Q^2} e^{Q^2}$$

$$u_n(Q) = \left(\frac{1}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}}$$

$$\times (-1)^n \left(e^{Q^2} \frac{d^n}{dQ^n} e^{-Q^2}\right) e^{-\frac{1}{2}Q^2}$$

$$H_n(Q)$$

$$u_n(Q) = \left(\frac{1}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(Q) e^{-\frac{1}{2}Q^2} \quad \checkmark$$

αx

$$Q = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x = \alpha x$$

$$\alpha = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}$$

$$\int_{-\infty}^{\infty} |u_n(Q)|^2 dQ = 1 \quad \checkmark$$

$$\int_{-\infty}^{\infty} |u_n(x)|^2 dx = 1$$

$$dQ = \alpha\, dx$$

$$\alpha \int_{-\infty}^{\infty} |u_n(\alpha x)|^2 dx = 1$$

$$\alpha^{\frac{1}{2}} u_n(\alpha x) \longrightarrow \underbrace{u_n(x)}_{\text{new (renormalized) } u_n}$$

$$u_n(x) = \left(\frac{\alpha}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2}$$

$$\int_{-\infty}^{\infty} |u_n(x)|^2 dx = 1$$

How do we calculate matrix elements of various operators using the eigenstates $|n\rangle$?

$$\hat{F} = F(\hat{x}, \hat{p}, t)$$ ✓

- relate $\hat{x}$ and $\hat{p}$ to $\hat{a}$ and $\hat{a}^\dagger$
- calculate $\langle n|\hat{F}|n'\rangle$ using

$$a|n\rangle = n^{\frac{1}{2}}|n-1\rangle$$

$$a^\dagger|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$$

$$Q = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x, \quad P = \frac{p}{(\hbar\omega m)^{\frac{1}{2}}}$$

$$Q = \frac{1}{\sqrt{2}}(a+a^\dagger)$$

$$P = \frac{1}{i\sqrt{2}}(a-a^\dagger)$$

$$x = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}} Q = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2}}$$

$$\times (a+a^\dagger)$$

$$x = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a+a^\dagger)$$

$$p = (\hbar\omega m)^{\frac{1}{2}} P = (\hbar\omega m)^{\frac{1}{2}} \frac{1}{i\sqrt{2}}$$

$$\times (a-a^\dagger)$$

$$p = -i\left(\frac{\hbar\omega m}{2}\right)^{\frac{1}{2}}(a-a^\dagger)$$

$$p = i\left(\frac{\hbar \omega m}{2}\right)^{\frac{1}{2}}(a^\dagger - a)$$

Example:

$$\langle n|V|n\rangle, \quad V = \frac{m\omega^2}{2}x^2$$

$$\langle n|\frac{m\omega^2}{2}x^2|n\rangle = \frac{m\omega^2}{2}$$

$$\times \langle n|\frac{\hbar}{2m\omega}(a+a^\dagger)^2|n\rangle$$

$$= \frac{\cancel{m}\cancel{\omega^2}\,\omega}{2}\;\frac{\hbar}{2\cancel{m}\cancel{\omega}}\langle n|(a+a^\dagger)^2|n\rangle$$

$$= \frac{1}{4}\hbar\omega\langle n|(a+a^\dagger)^2|n\rangle$$

$$= \frac{1}{4}\hbar\omega\langle n|\,a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a|n\rangle$$

$$a|n\rangle = n^{\frac{1}{2}}|n-1\rangle \quad \checkmark$$

$$a^2|n\rangle = [n(n-1)]^{\frac{1}{2}}|n-2\rangle$$

$$\sim |n-2\rangle$$

$$\langle n|a^2|n\rangle \sim \langle n|n-2\rangle = 0$$

$$a^{+}|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle \quad \checkmark$$

$$(a^{+})^2|n\rangle = [(n+1)(n+2)]^{\frac{1}{2}}|n+2\rangle$$

$$\sim |n+2\rangle$$

$$\langle n|(a^{+})^2|n\rangle \sim \langle n|n+2\rangle = 0$$

$$\langle n|V|n\rangle = \frac{1}{4}\hbar\omega\langle n|aa^{+}+a^{+}a|n\rangle$$

$$a^{+}a = N, \quad aa^{+} = a^{+}a + 1 = N+1$$

$$\langle n|V|n\rangle = \frac{1}{4}\hbar\omega\langle n|\,N+1+N\,|n\rangle$$
$$= \frac{1}{4}\hbar\omega\langle n|2N+1|n\rangle$$

$$N|n\rangle = n|n\rangle$$

$$\langle n|V|n\rangle = \frac{1}{4}\hbar\omega(2n+1)\underbrace{\langle n|n\rangle}_{=1}$$

$$\langle n|V|n\rangle = \frac{1}{4}\hbar\omega(2n+1)$$
$$= \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right)$$
$$= \frac{1}{2}E_n$$

Last example:

$$\langle n | x | n+1 \rangle$$

$$\langle n | x | n+1 \rangle = \left(\frac{\hbar}{2m\omega} \right)^{\frac{1}{2}}$$

$$\times \langle n | a + a^{\dagger} | n+1 \rangle$$

$$a^{\dagger} | n+1 \rangle \sim | n+2 \rangle$$

$$\langle n | a^{\dagger} | n+1 \rangle \sim \langle n | n+2 \rangle = 0$$

$$\langle n | x | n+1 \rangle = \left(\frac{\hbar}{2m\omega} \right)^{\frac{1}{2}} \langle n | a | n+1 \rangle$$

$$a | n+1 \rangle = (n+1)^{\frac{1}{2}} | n \rangle$$

$$\langle n | x | n+1 \rangle = \left(\frac{\hbar}{2m\omega} \right)^{\frac{1}{2}} (n+1)^{\frac{1}{2}} \underbrace{\langle n | n \rangle}_{1}$$

$$\langle n|x|n+1\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(n+1)^{\frac{1}{2}}$$

$$= \left[\frac{\hbar(n+1)}{2m\omega}\right]^{\frac{1}{2}}$$

$$\alpha = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}$$

$$\langle n|x|n+1\rangle = \frac{1}{\alpha}\left(\frac{n+1}{2}\right)^{\frac{1}{2}}$$

$$\langle n|x|n\rangle \sim \langle n|a+a^\dagger|n\rangle = 0,$$

$$\langle n|p|n\rangle \sim \langle n|a-a^\dagger|n\rangle = 0,$$

etc.

THE END

In a theory of many-body systems, we can follow similar ideas.

For example, for many-electron systems,

single-particle states

$$H = \sum_{p,q} \langle p | \hat{z} | q \rangle a_p^\dagger a_q + \frac{1}{2} \sum_{p,q,r,s} \langle pq | \hat{v} | rs \rangle \times a_p^\dagger a_q^\dagger a_s a_r$$

$$\hat{z}(i) = -\frac{\hbar^2}{2m_e} \nabla_i^2 - \sum_A \frac{Z_A e^2}{r_{iA}}$$

$$\hat{v}(ij) = \frac{e^2}{r_{ij}}$$

$$\hat{H} = \sum_{i=1}^{N} \hat{z}(i) + \sum_{1=i<j}^{N} \hat{v}(ij)$$