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DIRAC's δ "FUNCTION"

$$\{u_i(\underbrace{q_1,\ldots,q_f}_{q})\}_{i=1}^{\infty} \longrightarrow \text{basis in } \mathcal{H} = L^2(\Gamma)$$

$$\sum_i u_i^*(q')\, u_i(q) = \delta(q'-q)$$

closure relation

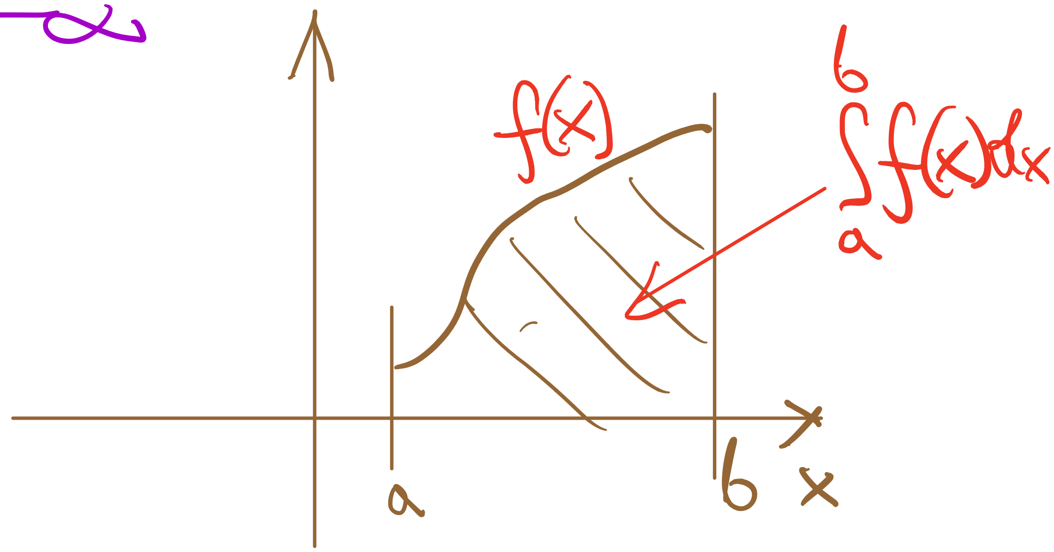
$$\delta(q'-q) = \prod_{l=1}^{f} \delta(q_l'-q_l)$$

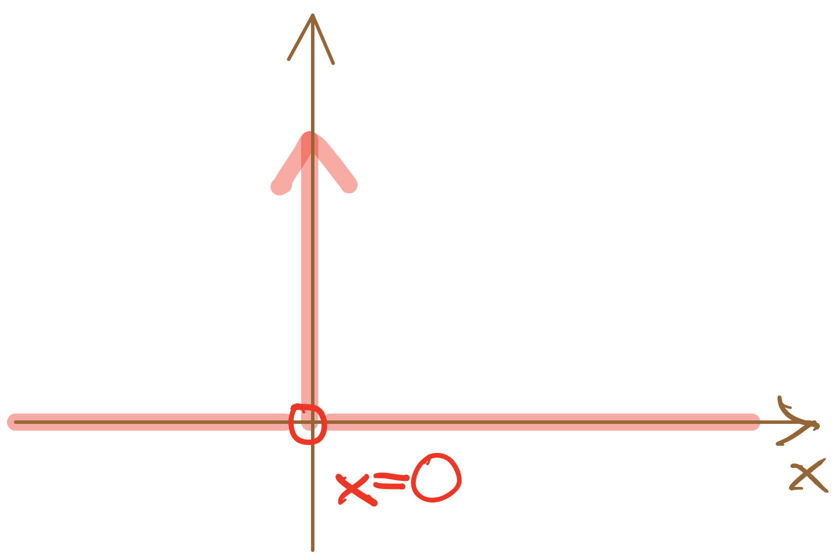
$\delta(q_l'-q_l) \leftarrow$ Dirac's δ function

$$\sum_i |u_i\rangle\langle u_i| = \mathbb{1}$$ (also closure)

$$\delta(x) = 0, \quad x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$





δ is not a normal function

δ is a DISTRIBUTION
(Schwartz)

Distributions are linear functionals defined on sufficiently well-behaved functions:

$$D[f] \in \mathbb{R}, \quad f = f(\underline{x}), \quad \underline{x} \in \mathbb{R}^n$$

$$\underline{x} = (x_1, \dots, x_n)$$

$$D[\alpha_1 f_1 + \alpha_2 f_2] = \alpha_1 D[f_1] + \alpha_2 D[f_2]$$

Dirac's delta (δ) is a distribution defined as

$$\boxed{\delta[f] = f(0).} \leftarrow f(x=0)$$

$$\delta[\alpha_1 f_1 + \alpha_2 f_2] = \alpha_1 f_1(0) + \alpha_2 f_2(0)$$
$$= \alpha_1 \delta[f_1] + \alpha_2 \delta[f_2]$$

There is large class of distributions defined as

$$D[f] = \int_{\mathbb{R}^n} \underbrace{K(\underline{x})}_{\text{kernel}} f(\underline{x})\, d\underline{x} \quad ✓ \qquad (1)$$

Dirac's δ IS NOT ONE OF THEM, but $\exists\, K_n(x)$

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} K_n(x) f(x)\, dx = \delta[f]$$
$$= f(0) \quad ✓$$

δ is not given by Eq. (1), since there is no function $K(x)$, such that

$$\int_{-\infty}^{\infty} K(x) f(x)\, dx = f(0).$$

$K(x)$ would have to be 0 for $x \neq 0$ and, at the same time,

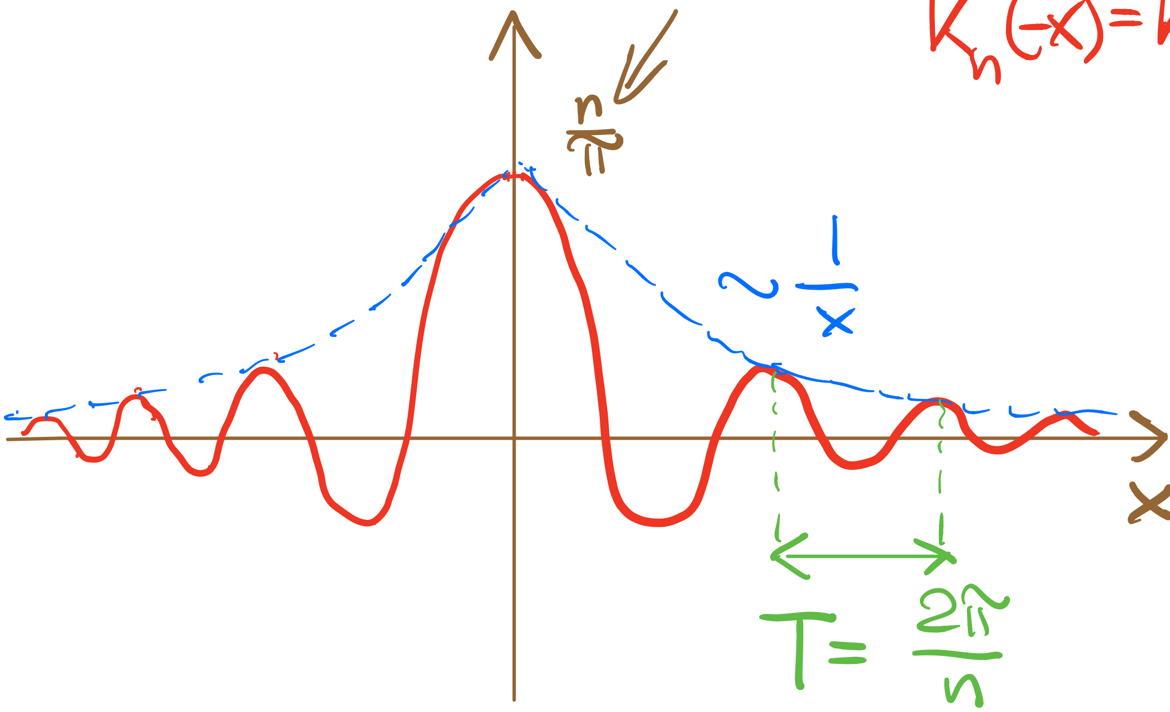
$$\int_{-\infty}^{\infty} K(x)\, dx = 1.$$

To obtain δ, we define

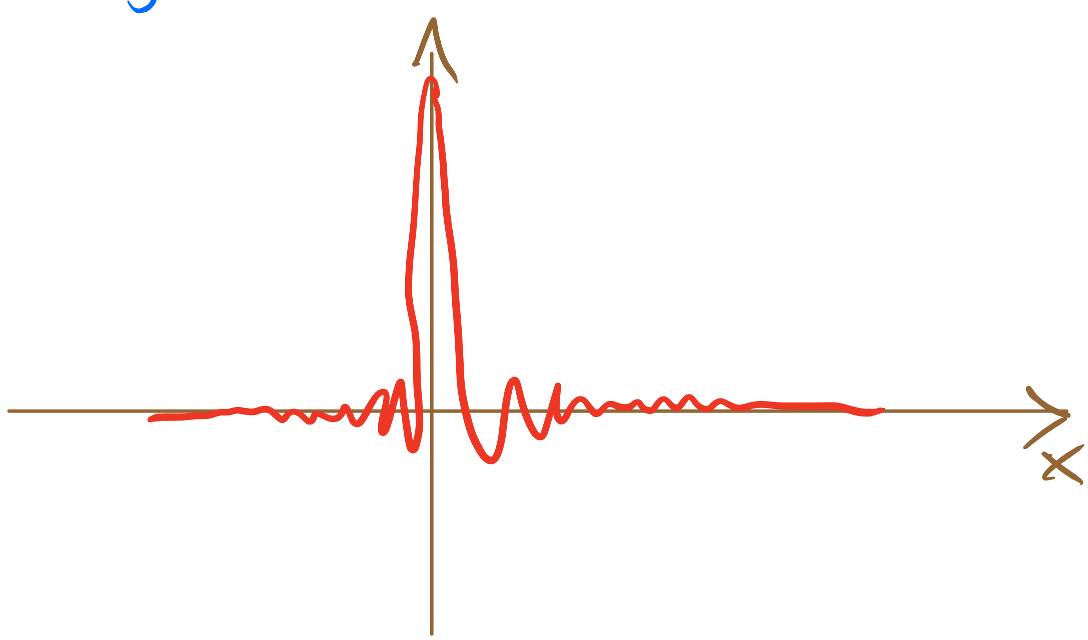
$$K_n(x) = \frac{\sin nx}{\pi x}.$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(x) f(x)\, dx = f(0)$$
$$= \delta[f]$$

$K_n(-x) = K_n(x)$



As $n \to \infty$, $K_n(x) \longrightarrow 0$ ✓
for $x \neq 0$ and $K_n(x=0) \to \infty$



Furthermore, $\forall_n$

$$\int_{-\infty}^{\infty} K_n(x)\,dx = \int_{-\infty}^{\infty} \frac{\sin nx}{\pi x}\,dx = 1 \quad \checkmark$$

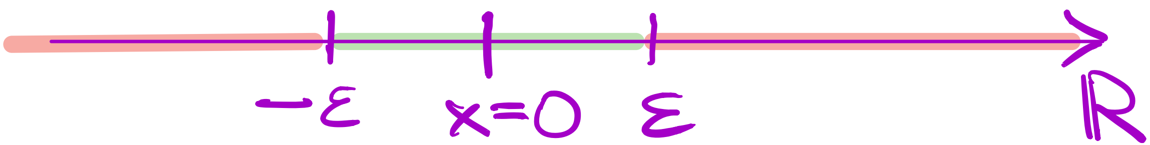
$$\int_{-\infty}^{\infty} \frac{\sin y}{y}\,dy = \pi$$

$$y = nx \,,\; dy = n\,dx$$

$$\int_{-\infty}^{\infty} \frac{\sin nx}{\cancel{n}x}\,\cancel{n}\,dx = \pi \;/ :\pi$$

$$\int_{-\infty}^{\infty} \frac{\sin nx}{\pi x}\,dx = 1 \quad \checkmark$$

$$\int_{-\infty}^{\infty} K_n(x) f(x)\,dx = \underbrace{\int_{-\infty}^{-\varepsilon} K_n(x) f(x)\,dx}_{\xrightarrow{n\to\infty} 0}$$



$$+ \int_{-\varepsilon}^{\varepsilon} K_n(x) f(x)\,dx$$

$$+ \underbrace{\int_{\varepsilon}^{\infty} K_n(x) f(x)\,dx}_{\xrightarrow{n\to\infty} 0}$$

$$\xrightarrow[n\to\infty]{} \int_{-\varepsilon}^{\varepsilon} K_n(x) f(x)\,dx$$

$$\underset{\substack{n\to\infty \\ \varepsilon\to 0}}{\simeq} f(0)\int_{-\varepsilon}^{\varepsilon} K_n(x)\,dx$$

$$\underset{n\to\infty}{\simeq} f(0)\underbrace{\int_{-\infty}^{\infty} K_n(x)\,dx}_{1}$$

$$= f(0)$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} K_n(x)f(x)\,dx = f(0) = \delta[f] \quad \checkmark$$

We can think of δ as follows:

$$\int_{-\infty}^{\infty} \delta(x)f(x)\,dx = f(0) \quad \checkmark$$

(symbolically)

$$\delta(x) = \lim_{n\to\infty} K_n(x)$$ ✓✓

$$\delta(x) = \lim_{n\to\infty} \frac{\sin nx}{\pi x}$$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,dx \text{ "="}$$

$$\text{"="} \int_{-\infty}^{\infty} \left(\lim_{n\to\infty} K_n(x)\right) f(x)\,dx$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(x) f(x)\,dx$$

$$= f(0) = \delta[f]$$

distribution

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0) = \delta[f]$$

"function"

✓✓✓✓

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1 \quad \checkmark$$

$$\lim_{n\to\infty} K_n(x) = \delta(x)$$

$$\delta(x) = 0 \,, \; x \neq 0 \quad \checkmark\checkmark$$

$$\int_a^b \delta(x)\,dx = 1 \quad \text{if} \quad a < 0 < b$$

$$\int_a^b \delta(x) f(x)\,dx = f(0) \qquad a < 0 < b$$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0) \quad \checkmark$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx$$

$$\underset{dy=dx}{\overset{y=x-a}{=\!=}} \int_{-\infty}^{\infty} f(y+a)\,\delta(y)\,dy$$

$$= f(a)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a) \quad \checkmark$$

Closure relation:

$$\varphi(q_1,\dots,q_f) \equiv \varphi(q)$$

$$\varphi(q) = \sum_i c_i u_i(q) \quad \checkmark$$

$$\{u_i(q)\}_{i=1}^{\infty} \longrightarrow \text{orthonormal basis}$$

$$c_i = \langle u_i | \varphi \rangle$$

$$= \int_{\Gamma} u_i^*(q') \, \varphi(q') \, d\tau'$$

$$\varphi(q) = \sum_i u_i(q) \int_{\Gamma} u_i^*(q') \varphi(q') d\tau'$$

$$= \int_{\Gamma} \varphi(q') \underbrace{\left(\sum_i u_i^*(q') u_i(q) \right)}_{\delta(q'-q)} d\tau'$$

$$\int_{\Gamma} \varphi(q') \, \delta(q'-q) \, d\tau' = \varphi(q)$$

$$\sum_i u_i^*(q') u_i(q) = \delta(q'-q)$$

$$\delta(q'-q) = \prod_{l=1}^{f} \delta(q_l' - q_l)$$

Why is this called closure?

$$\langle \varphi | \psi \rangle = \int_\Gamma \varphi^*(q)\,\psi(q)\,d\tau$$

$$= \int_\Gamma \int_\Gamma \varphi^*(q)\,\delta(q'-q)\,\psi(q')\,d\tau'\,d\tau$$

$$= \int_\Gamma \int_\Gamma \varphi^*(q) \underbrace{\left[\sum_i u_i^*(q')\,u_i(q)\right]}_{\delta(q'-q)}$$

$$\times\ \psi(q')\,d\tau'\,d\tau$$

$$\langle \varphi | \psi \rangle = \sum_i \int_\Gamma \varphi^*(q)\, u_i(q)\, d\tau$$

$$\times \int_\Gamma u_i^*(q')\, \psi(q')\, d\tau'$$

$$= \sum_i \langle \varphi | u_i \rangle \langle u_i | \psi \rangle$$

$\therefore$

$$\sum_i |u_i\rangle\langle u_i| = \mathbb{1}$$

is equivalent to

$$\sum_i u_i^*(q')\, u_i(q) = \delta(q'-q).$$

EXPECTATION VALUE (REVISITED)

$\varphi, \quad \|\varphi\| = 1,$

$$\langle \varphi | \varphi \rangle = \|\varphi\|^2 = 1.$$

$\hat{F}$, self-adjoint operator with discrete (point) spectrum

normalized

$$\hat{F} |f_\mu\rangle = \lambda_\mu |f_\mu\rangle$$

f_μ's form a basis in $\mathcal{H}$

$$|\varphi\rangle = \sum_{\mu} c_{\mu} |f_{\mu}\rangle, \quad \varphi \in \mathcal{H}$$

$$c_{\mu} = \langle f_{\mu} | \varphi \rangle$$

$$\sum_{\mu} |c_{\mu}|^2 = 1$$

$$1 = \langle \varphi | \varphi \rangle = \left\langle \sum_{\mu} c_{\mu} f_{\mu} \middle| \sum_{\nu} c_{\nu} f_{\nu} \right\rangle$$

$$= \sum_{\mu,\nu} c_{\mu}^{*} c_{\nu} \underbrace{\langle f_{\mu} | f_{\nu} \rangle}_{\delta_{\mu\nu}}$$

$$= \sum_{\mu} c_{\mu}^{*} c_{\mu} = \sum_{\mu} |c_{\mu}|^2$$

We interpret $|c_\mu|^2$ as the probability that the value of F is state $|\varphi\rangle$ is λ_μ,

$$P(F=\lambda_\mu)=|c_\mu|^2.$$

The expectation value of F is

$$\langle\hat{F}\rangle=\sum_\mu \lambda_\mu\, P(F=\lambda_\mu),$$

$$\sum_\mu P(F=\lambda_\mu)=1$$

$$P(F=\lambda_\mu)=|c_\mu|^2$$

Previously,

$$\langle \hat{F} \rangle = \langle \varphi | \hat{F} | \varphi \rangle$$

$$= \int \varphi^* \hat{F} \varphi \, d\tau$$

$$\|\varphi\| = 1.$$

$$\langle \hat{F} \rangle = \sum_\mu \lambda_\mu P(F = \lambda_\mu)$$

$$= \sum_\mu \lambda_\mu |c_\mu|^2$$

$$= \sum_\mu \lambda_\mu c_\mu^* c_\mu$$

$$= \sum_\mu \lambda_\mu \langle f_\mu | \varphi \rangle^* \langle f_\mu | \varphi \rangle$$

$$= \sum_\mu \lambda_\mu \langle \varphi | f_\mu \rangle \langle f_\mu | \varphi \rangle$$

$$= \sum_\mu \langle \varphi | \underbrace{\lambda_\mu f_\mu}_{\hat{F} | f_\mu \rangle} \rangle \langle f_\mu | \varphi \rangle$$

$$\overset{\text{TOR}}{=} \sum_\mu \langle \varphi | \hat{F} f_\mu \rangle \langle f_\mu | \varphi \rangle$$

$$= \sum_\mu \langle \hat{F} \varphi | f_\mu \rangle \langle f_\mu | \varphi \rangle$$

$$\sum_\mu | f_\mu \rangle \langle f_\mu | = \mathbb{1}$$

$$= \langle \hat{F} \varphi | \varphi \rangle \overset{\text{TOR}}{=} \langle \varphi | \hat{F} \varphi \rangle$$

$$\langle \hat{F} \rangle = \langle \varphi | \hat{F} \varphi \rangle$$

$$= \langle \varphi | \hat{F} | \varphi \rangle \;\therefore$$

Another proof:

$$\langle \varphi | \hat{F} | \varphi \rangle = \left\langle \sum_{\mu} c_\mu f_\mu \middle| \hat{F} \middle| \sum_{\nu} c_\nu f_\nu \right\rangle$$

$$= \sum_{\mu,\nu} c_\mu^* c_\nu \langle f_\mu | \hat{F} | f_\nu \rangle$$

$$\langle f_\mu | \underbrace{\hat{F} | f_\nu}_{\lambda_\nu | f_\nu \rangle} \rangle = \lambda_\nu \langle f_\mu | f_\nu \rangle$$

$$= \lambda_\nu \delta_{\mu\nu} = \lambda_\mu \delta_{\mu\nu} \quad \checkmark$$

$$= \sum_{\mu,\nu} c_\mu^* c_\nu \lambda_\mu \delta_{\mu\nu}$$

$$= \sum_\mu c_\mu^* c_\mu \lambda_\mu$$

$$\langle \varphi | \hat{F} | \varphi \rangle = \sum_\mu \lambda_\mu |c_\mu|^2$$

$$= \sum_\mu \lambda_\mu P(F = \lambda_\mu)$$

$\therefore$

If $\|\varphi\| \neq 1$,

$$\langle \hat{F} \rangle = \frac{\langle \varphi | \hat{F} | \varphi \rangle}{\langle \varphi | \varphi \rangle}$$

✓ # CONTINUOUS SPECTRA

$\hat{H}$ of the harmonic oscillator has discrete spectrum only

$\hat{x}, \hat{p}_x$ have continuous spectra only.

$\hat{H}$ of the hydrogen atom has discrete and continuous spectra.

set

solve for this

✓ $$\hat{F}|f_\mu\rangle = \lambda_\mu |f_\mu\rangle \quad (2)$$

μ is a continuous variable

$\lambda_\mu = \lambda(\mu)$ [function of μ]

$f_\mu(q) = f(q;\mu)$

parameter

[function parametrized by μ]

$\mu \longrightarrow \mu + \Delta\mu$

$\lambda_\mu \longrightarrow \lambda_{\mu+\Delta\mu}$ also solves Eq. (2)

Example:

$\hat{p}_x$, Ae^{ikx}

$$\hat{p}_x e^{ikx} = \frac{\hbar}{i}\frac{\partial}{\partial x} e^{ikx}$$

$$= \frac{\hbar}{\cancel{i}} \cancel{i} k \, e^{ikx}$$

$$= \underbrace{(\hbar k)}_{\text{continuous}} e^{ikx}$$