ORTHONORMAL BASES

$\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis in the Hilbert space if $\forall_{i,j}$ $(e_i, e_j) = \delta_{ij}$ and $\{e_i\}_{i=1}^{\infty}$ is closed ($\bigoplus_{i=1}^{\infty} \langle e_i \rangle$ is dense in $\mathcal{H}$) and complete (the only vector orthogonal to all e_i's is Θ (zero vector)).

$$\boxed{(g,h) = \sum_{i=1}^{\infty} (g, e_i)(e_i, h)}$$

(generalized Parseval relation)

$$(h,h) = \|h\|^2 = \sum_{i=1}^{\infty} (h,e_i)(e_i,h)$$
$$= \sum_{i=1}^{\infty} |(h,e_i)|^2$$

(closure or Parseval relation)

QUANTUM MECHANICS

$$\mathcal{H} = L^2(\Gamma), \quad \int_\Gamma |\psi|^2 d\tau < \infty$$

Orthonormal basis: $\{u_i\}_{i=1}^{\infty}$, $(u_i,u_j) = \delta_{ij}$

$$\langle u_i | u_j \rangle = \delta_{ij}$$

$\bigoplus_{i=1}^{\infty} \langle u_i \rangle$ is dense in $L^2(\Gamma)$

$$\left\| \varphi - \sum_{i=1}^{n} c_i u_i \right\| \xrightarrow[n\to\infty]{} 0$$

$$\varphi = \sum_{i=1}^{\infty} c_i u_i, \quad c_i = (u_i, \varphi)$$

$$|\varphi\rangle = \sum_{i=1}^{\infty} c_i |u_i\rangle \quad \checkmark$$

$$c_i = \langle u_i|\varphi\rangle \quad \checkmark$$

$$|\varphi\rangle = \sum_{i=1}^{\infty} \langle u_i|\varphi\rangle |u_i\rangle$$

$$= \sum_{i=1}^{\infty} \underline{|u_i\rangle\langle u_i|}\varphi\rangle$$

$$\sum_{i=1}^{\infty} \underbrace{|u_i\rangle\langle u_i|}_{P_{\langle u_i\rangle}} = \mathbb{1} \leftarrow \text{the unit operator}$$

(resolution of identity)

$$\sum_{i=1}^{\infty} P_{\langle u_i\rangle} = \mathbb{1}$$

$$\bigoplus_{i=1}^{\infty} \langle u_i\rangle = \mathcal{H}$$

$$\sum_{i=1}^{\infty} P_{\langle u_i \rangle} = P_{\mathcal{H}} = \mathbb{1}$$

$$\begin{aligned}
(\varphi, \psi) &\equiv \langle \varphi | \psi \rangle \\
&= \langle \varphi | \mathbb{1} | \psi \rangle \\
&= \langle \varphi | \sum_{i=1}^{\infty} | u_i \rangle \langle u_i | \, | \psi \rangle \\
&= \sum_{i=1}^{\infty} \langle \varphi | u_i \rangle \langle u_i | \psi \rangle \\
&= \sum_{i=1}^{\infty} (\varphi, u_i)(u_i, \psi)
\end{aligned}$$

$$\varphi = \sum_{i=1}^{\infty} c_i u_i, \quad \psi = \sum_{i=1}^{\infty} d_i u_i$$

$$c_i = (u_i, \varphi), \quad d_i = (u_i, \psi)$$

$$(\varphi,\psi) = \sum_{i=1}^{\infty} \underbrace{(\varphi,u_i)}_{c_i^*}\underbrace{(u_i,\psi)}_{d_i}$$

$$(\varphi,\psi) = \sum_{i=1}^{\infty} c_i^* d_i$$

$$\langle\varphi|\psi\rangle = \sum_{i=1}^{\infty} c_i^* d_i$$

completeness relation (closure)

$$\varphi = \psi$$

$$\langle\varphi|\varphi\rangle = (\varphi,\varphi) = \|\varphi\|^2$$

$$= \sum_{i=1}^{\infty} c_i^* c_i$$

$$\|\varphi\|^2 = \sum_{i=1}^{\infty} |c_i|^2$$

$$c_i = (u_i,\varphi) = \langle u_i|\varphi\rangle$$

What if $\|\varphi\| = 1$?

$$\left(\int |\varphi|^2 d\tau = 1 \right)$$

$$1 = \|\varphi\|^2 = \sum_{i=1}^{\infty} |c_i|^2$$

$$\sum_{i=1}^{\infty} |c_i|^2 = 1 \quad \checkmark$$

Theorem: Let A be a self-adjoint operator whose spectrum consists of a point spectrum only (eigenvalues). There exists an orthonormal basis in $\mathcal{H}$ consisting entirely of eigenvectors of A.

Proof:

Let $\lambda_1, \lambda_2, \ldots$ be the distinct eigenvalues of A (all of them).

$$A f_{ip} = \lambda_i f_{ip}, \quad p = 1, \ldots, n_i$$

degeneracy

$$G_i = \mathrm{Span}\{f_{ip}, p = 1, \ldots, n_i\}$$

eigenspace corresponding to λ_i

For $i \neq j$ $(\lambda_i \neq \lambda_j)$,

$$G_i \perp G_j$$

From now on we will assume that $(f_{ip}, f_{iq}) = \delta_{pq}$.

$$(f_{ip}, f_{jq}) = \delta_{ij} \cdot \delta_{pq}$$

$\{f_{ip}\}_{i=1,\, p=1,\dots,n_i}^{\infty}$ is an orthonormal set.

√ We will prove now that $\{f_{ip}\}$ is an orthonormal basis.

Define

$$G = G_1 \oplus G_2 \oplus \cdots$$
$$= \bigoplus_i G_i \quad ✓$$

$$G \subseteq \mathcal{H}$$

$$\overline{G} = \langle \dots f_{ip} \dots \rangle$$

$$\forall_{g\in G} \quad Ag \in G \;\checkmark$$

$$g = g_1 + g_2 + \dots = \sum_i g_i$$

$$g_i \in G_i\,,\quad g_i \perp g_j\,,\; i \neq j$$

$$Ag = A\sum_i \underbrace{g_i}_{\in G_i} = \sum_i \underbrace{Ag_i}_{\lambda_i g_i}$$

$$Ag = \sum_i \underbrace{\lambda_i g_i}_{\in G_i} \in G \;\checkmark$$

The main part of the proof:

We want to prove that $\{f_{ip}\}$ forms a basis.

We must prove that $\{f_{i\varphi}\}$ is complete. In other words, we must prove that there exists no nonzero vector from $\mathcal{H}$ which is orthogonal to all $f_{i\varphi}$'s.

Proof by contradiction:

$\dim G^{\perp} > 0$

Suppose $G^{\perp} \neq \{\Theta\}$

(suppose there exists $h \in \mathcal{H}, h \neq \Theta$ orthogonal to G)

$$\underset{\substack{h \in \mathcal{H} \\ h \neq \Theta}}{\exists} \quad (h, g) = 0 \quad \underset{g \in G}{\forall}$$

Clearly, h must belong to $G^{\perp}$,

$h \in G^{\perp}$

$$0 = (\underset{\underset{G^\perp}{\cap}}{h}, A\underset{\underset{G}{\cap}}{g}) \overset{\text{TOR}}{\underset{A=A^\dagger}{=}} (Ah, g)$$

$$Ah \perp g \in G \quad \forall_{g \in G}$$

This means that

$$Ah \in G^\perp \text{ for } h \in G^\perp.$$

Define

$\tilde{A}$: A restricted to $G^\perp$, $\tilde{A} = A|_{G^\perp}$. ✓

$$\tilde{A} : G^\perp \to G^\perp$$

$$\tilde{A}h = Ah, \quad h \in G^\perp.$$

Can we solve $\tilde{A}f = \lambda f$ in $G^\perp$, $f \in G^\perp$, $f \neq 0$?

The answer is YES. ✓

Here is an explanation:

Consider B defined in finite-dimensional space V.

$$Bf = \lambda f \quad \checkmark$$

$$\underline{\underline{B}} \cdot \underline{f} = \lambda \underline{f} \quad \checkmark \quad \left(\text{in matrix form}\right)$$

$$\begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \cdots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \lambda \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \quad \checkmark$$

$$\left(\underline{\underline{B}} - \lambda \underline{\underline{1}}\right) \cdot \underline{f} = 0 \quad \checkmark$$

$$\begin{pmatrix} B_{11}-\lambda & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22}-\lambda & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ B_{n1} & \cdots & & B_{nn}-\lambda \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \checkmark$$

We are looking for $f \neq \Theta$,

$$\det(\underline{\underline{B}} - \lambda \underline{\underline{1}}) = 0.$$

secular equation (polynomial of order n for λ).

There exists at least one λ that solves it.

This means that

$\hat{A}f = \lambda f$ has a solution, λ, $f \neq \Theta$, in

$G^\perp$. Since $\tilde{A} = A|_{G^\perp}$, this also means that

$$Af = \lambda f \text{ has}$$

a solution λ, $f \neq \Theta$ in $G^\perp$ $(f \in G^\perp)$.

This is a contradiction with an assumption that $\lambda_1, \lambda_2, \ldots$ are all eigenvalues of A. This means that the assumption that $G^\perp \neq \{\Theta\}$ was wrong, $G^\perp = \{\Theta\}$, and $\{f_{ip}\}$ forms a complete set.

$\therefore$

SPECTRAL THEOREMS

Let P_{G_i} be a projection operator (projector) on G_i, where G_i is an eigenspace of operator A corresponding to eigenvalue λ_i.

$$\bigoplus_i G_i = \mathcal{H}$$

$$\sum_{i=1}^{\infty} P_{G_i} = \mathbb{1} \quad \checkmark$$

$$A P_{G_i} = \lambda_i P_{G_i} \quad \checkmark$$

$$P_{G_i} h \in G_i \quad \forall_{h \in \mathcal{H}}$$

$$(A P_{G_i}) h = A(\underbrace{P_{G_i} h}_{\in G_i}) \quad \forall_{h \in \mathcal{H}}$$

$(P_{G_i} h$ is a linear combination of f_{ip}'s with $p = 1, \ldots, n_i)$

$$= \lambda_i (P_{G_i} h)$$

$$A P_{G_i} = \lambda_i P_{G_i}$$

$$A = A \mathbb{1} = A \sum_i P_{G_i}$$

$$= \sum_i A P_{G_i} = \sum_i \lambda_i P_{G_i}$$

spectral theorem for A

$$\boxed{A = \sum_i \lambda_i P_{G_i}} \quad \checkmark\checkmark$$

$$\left(\hat{H} = \sum_{\mu} E_{\mu} P_{G_{\mu}}\right)$$

assuming point or discrete spectrum only

In QM:

distinct eigenvalues

$$\hat{F}\,|f_{\mu p}\rangle = \lambda_{\mu}\,|f_{\mu p}\rangle$$

eigenstates of $\hat{F}$

$p = 1, \dots, n_{\mu}$

$\{f_{\mu p}\} \rightarrow$ orthonormal basis

$$|\varphi\rangle = \sum_{\mu} \sum_{p=1}^{n_{\mu}} c_{\mu p}\,|f_{\mu p}\rangle$$

$$c_{\mu p} = \langle f_{\mu p} | \varphi \rangle$$

eigenspace corresponding to λ_{μ}

$$G_{\mu} = \mathrm{Span}\{f_{\mu p},\ p = 1, \dots, n_{\mu}\}$$

$\checkmark$ $$P_{G_\mu} = \sum_{p=1}^{n_\mu} \underbrace{|f_{\mu p}\rangle\langle f_{\mu p}|}_{P_{\langle f_{\mu p}\rangle}}$$

$$\sum_\mu P_{G_\mu} = \mathbb{1}$$

$$\hat{F} = \hat{F}\mathbb{1} = \sum_\mu \hat{F} P_{G_\mu}$$

$$= \sum_\mu \hat{F} \sum_{p=1}^{n_\mu} |f_{\mu p}\rangle\langle f_{\mu p}|$$

$$= \sum_\mu \sum_{p=1}^{n_\mu} \underbrace{\hat{F}|f_{\mu p}\rangle}_{\lambda_\mu |f_{\mu p}\rangle}\langle f_{\mu p}|$$

$$= \sum_\mu \sum_{p=1}^{n_\mu} \lambda_\mu |f_{\mu p}\rangle\langle f_{\mu p}|$$

$$\hat{F} = \sum_{\mu} \lambda_\mu \underbrace{\sum_{p=1}^{n_\mu} |f_{\mu p}\rangle\langle f_{\mu p}|}_{P_{G_\mu}}$$

$$\hat{F} = \sum_{\mu} \lambda_\mu P_{G_\mu}$$

spectral theorem for $\hat{F}$.

We could repeat this without emphasizing degeneracies:

basis

$$\hat{F}|f_\mu\rangle = \lambda_\mu |f_\mu\rangle$$

(repetions among λ_μ allowed)

$$|\varphi\rangle = \sum_\mu c_\mu |f_\mu\rangle$$

$$c_\mu = \langle f_\mu | \varphi\rangle$$

$$\hat{F} = \hat{F}\,\mathbb{1}$$

$$\sum_\mu |f_\mu\rangle\langle f_\mu| = \mathbb{1} \quad \checkmark$$

$$\hat{F} = \sum_\mu \underbrace{\hat{F}|f_\mu\rangle}_{f_\mu |f_\mu\rangle}\langle f_\mu|$$

$$\hat{F} = \sum_\mu f_\mu |f_\mu\rangle\langle f_\mu|$$

We must talk about Dirac's delta "function".

$$\{u_i(\underbrace{q_1,\ldots,q_f}_{q})\}_{i=1}^{\infty} \longrightarrow \text{basis in } \mathcal{H} = L^2(\Gamma)$$

$$\sum_i u_i^*(q')\, u_i(q) = \delta(q'-q)$$ closure relation

$$\delta(q'-q) = \prod_{l=1}^{f} \delta(q'_l - q_l)$$

$\delta(q'_l - q_l)$ is a Dirac delta

$$\sum_i |u_i\rangle\langle u_i| = \mathbb{1}$$ (also closure)