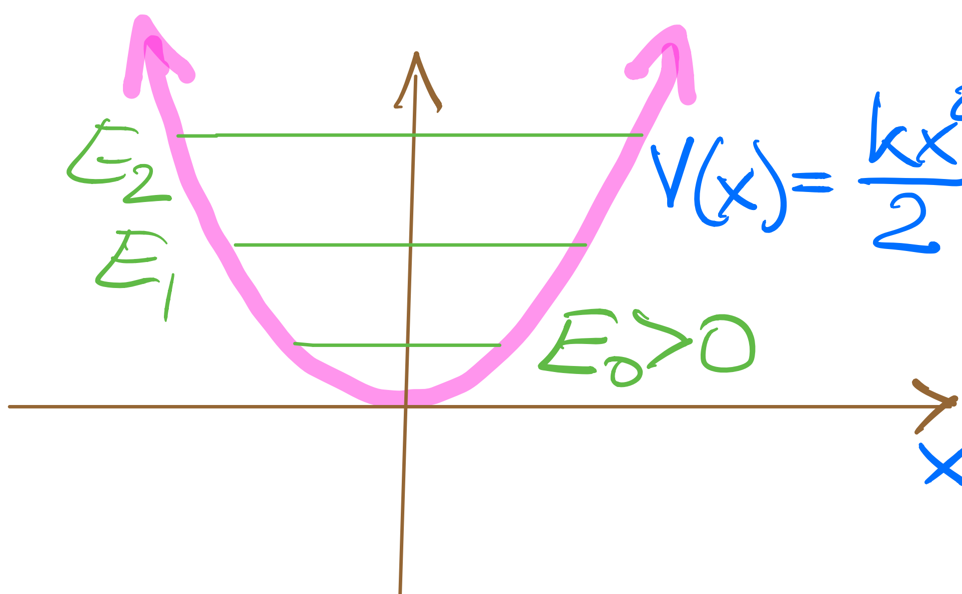
LINEAR HARMONIC OSCILLATOR

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2}+\frac{1}{2}kx^2u=Eu$$

$$u=u(x),\ k=m\omega^2$$



$$V(x)=\frac{kx^2}{2}$$

We anticipate only discrete spectrum (bound states)

DIMENSIONLESS FORM OF THE SCHRÖDINGER EQUATION

$$\boxed{\begin{gathered}\frac{d^2u}{d\xi^2}+(\lambda-\xi^2)u=0\\ \xi=\alpha x\\ \alpha=\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}\\ \lambda=\frac{2E}{\hbar\omega}\end{gathered}}$$

THE SOMMERFELD POLYNOMIAL METHOD

Asymptotic solutions ($|\xi| \to \infty$)

$$u \sim \xi^n e^{\pm\frac{1}{2}\xi^2} \quad \checkmark$$

Proof:

$$u'' + (\lambda - \xi^2)u = 0$$

$$u'' = \frac{d^2u}{d\xi^2} = n(n-1)\xi^{n-2} e^{\pm\frac{1}{2}\xi^2}$$

$$+ 2n\xi^{n-1}\left(e^{\pm\frac{1}{2}\xi^2}\right)'$$

$$+ \xi^n\left(e^{\pm\frac{1}{2}\xi^2}\right)''$$

$$(fg)'' = f''g + 2f'g' + fg'' \quad \checkmark$$

$$\left(e^{\pm\frac{1}{2}\xi^2}\right)' = \pm\frac{1}{\cancel{2}}\cancel{2}\xi e^{\pm\frac{1}{2}\xi^2}$$

$$= \pm\xi e^{\pm\frac{1}{2}\xi^2} \quad \checkmark$$

$$\left(e^{\pm\frac{1}{2}\xi^2}\right)'' = \pm\, e^{\pm\frac{1}{2}\xi^2} + \xi^2 e^{\pm\frac{1}{2}\xi^2} \quad \checkmark$$

$$u'' = n(n-1)\xi^{n-2} e^{\pm\frac{1}{2}\xi^2}$$

$$\pm\, 2n\,\xi^n e^{\pm\frac{1}{2}\xi^2}$$

$$\pm\, \xi^n e^{\pm\frac{1}{2}\xi^2} + \xi^{n+2} e^{\pm\frac{1}{2}\xi^2}$$

$$u'' = \left[n(n-1)\xi^{n-2} \pm (2n+1)\xi^n + \xi^{n+2}\right] e^{\pm\frac{1}{2}\xi^2}$$

asymptotically,

$$u'' \approx \xi^{n+2} e^{\pm\frac{1}{2}\xi^2}$$

$$u'' + (\lambda - \xi^2)u \approx u'' - \xi^2 u$$

$$\approx \xi^{n+2} e^{\pm\frac{1}{2}\xi^2} - \xi^{n+2} e^{\pm\frac{1}{2}\xi^2} = 0$$

$$u = A\xi^m e^{+\frac{1}{2}\xi^2} + B\xi^n e^{-\frac{1}{2}\xi^2}$$

$$A\xi^m e^{+\frac{1}{2}\xi^2} \xrightarrow{|\xi|\to\infty} \infty$$

We must reject the first term.

$$u(\xi) \xrightarrow[|\xi|\to\infty]{} B\xi^n e^{-\frac{1}{2}\xi^2}$$

For finite ξ:

polynomial

$$u(\xi) = H(\xi) e^{-\frac{1}{2}\xi^2}$$

$$u'' + (\lambda - \xi^2)u = 0 \quad \checkmark$$

$$u'' = H'' e^{-\frac{1}{2}\xi^2} + 2H'\left(-\xi e^{-\frac{1}{2}\xi^2}\right) + H\left(-e^{-\frac{1}{2}\xi^2} + \xi^2 e^{-\frac{1}{2}\xi^2}\right)$$

$$u'' + (\lambda - \xi^2)u = \Big(H'' - 2\xi H' - H + \cancel{\xi^2 H} + \lambda H - \cancel{\xi^2 H}\Big) \times e^{-\frac{1}{2}\xi^2} = 0$$

$$H'' - 2\xi H' + (\lambda - 1)H = 0 \quad \checkmark$$

Let us assume that $H(\xi)$ is a series

$$H(\xi) = \xi^s(a_0 + a_1\xi + a_2\xi^2 + \dots)$$

$$s \geq 0, \; a_0 \neq 0$$

$$H(\xi) = \sum_{\nu=0}^{\infty} a_\nu \xi^{\nu+s} \quad \checkmark$$

It is convenient to write

$$H(\xi) = \sum_{\nu} a_\nu \xi^{\nu+s}$$

with an understanding that

$$a_{-1} = a_{-2} = a_{-3} = \dots = 0$$

$$a_0 \neq 0, \quad s \geq 0$$

$$H'' - 2\xi H' + (\lambda - 1) H = 0$$

$$\sum_{\nu} a_\nu (\nu+s)(\nu+s-1)\xi^{\nu+s-2} \qquad \nu \to \nu+2$$

$$- 2 \sum_{\nu} a_\nu (\nu+s) \xi^{\nu+s}$$

$$+ \sum_{\nu} (\lambda - 1) a_\nu \xi^{\nu+s} = 0$$

$$\sum_{\nu}\left\{a_{\nu+2}(\nu+s+2)(\nu+s+1) - (2\nu+2s+1-\lambda)a_{\nu}\right\}\xi^{\nu+s} = 0$$

$$a_{\nu+2}(\nu+s+2)(\nu+s+1) - (2\nu+2s+1-\lambda)a_{\nu} = 0 \quad \forall \nu \qquad (\star)$$

$$a_{-1} = a_{-2} = \cdots = 0$$

$$a_0 \neq 0, \quad s \geqslant 0$$

When $\nu < -2$, Eq. $(\star)$ is trivially satisfied.
In analyzing Eq. $(\star)$ we need to focus on $\nu \geqslant -2$.

$$\nu=-2:\quad a_0\, s(s-1)=0 \quad \checkmark$$

$$\nu=-1:\quad a_1(s+1)s=0 \quad \checkmark$$

$$\nu=0:\quad a_2(s+2)(s+1)$$
$$-(2s+1-\lambda)a_0=0 \quad \checkmark$$

$$\nu=1:\quad a_3(s+3)(s+2)$$
$$-(2s+3-\lambda)a_1=0 \quad \checkmark$$

$$\cdots$$

In general,

$$a_{\nu+2}(\nu+s+2)(\nu+s+1) \quad \checkmark$$
$$-(2\nu+2s+1-\lambda)a_\nu=0$$

From $a_0\, s(s-1)=0$ $(a_0\neq 0)$,
$s=0$ or $s=1$.

From $a_1(s+1)s = 0$,

$s = 0$ or $a_1 = 0$ or both.

From the remaining equations

$$a_0 \to a_2 \to a_4 \to a_6 \to \cdots$$

$$a_1 \to a_3 \to a_5 \to a_7 \to \cdots$$

In general, from a_ν we get $a_{\nu+2}$.

Since

$$V(x) = \frac{1}{2}kx^2$$

is even $[V(x) = V(-x)]$,

we can assume that

$u(-x) = u(x)$ (even) or

$u(-x) = -u(x)$ (odd)

$$u = H e^{-\frac{1}{2}\xi^2}$$

$$= H \underbrace{e^{-\frac{1}{2}\alpha^2 x^2}}_{\text{even}}$$

This means that H is either even, $H(-\xi) = H(\xi)$, or odd, $H(-\xi) = -H(\xi)$.

Even $H(\xi)$:

$$H(\xi) = H(-\xi)$$

$$\cancel{\xi^s}(a_0 + a_1\xi + a_2\xi^2 + \dots)$$

$$= (-1)^s \cancel{\xi^s}(a_0 - a_1\xi + a_2\xi^2 - \dots)$$

$$\cancel{a_0} = (-1)^s \cancel{a_0} \quad (a_0 \neq 0)$$

$\checkmark \; a_1 = (-1)^{s+1} a_1$, etc.

$$(-1)^s = 1 \Rightarrow s = 0.$$

$$a_1 = -a_1 \Rightarrow a_1 = 0.$$

a_ν with odd ν must be 0.

$$H(\xi) = a_0 + a_2 \xi^2 + \ldots$$

Odd $H(\xi)$:

$$H(\xi) = -H(-\xi)$$

$$\cancel{\xi^s}(a_0 + a_1\xi + a_2\xi^2 + \ldots)$$

$$= (-1)^{s+1} \cancel{\xi^s}(a_0 - a_1\xi + a_2\xi^2 - \ldots)$$

$$\cancel{a_0} = (-1)^{s+1} \cancel{a_0} \quad (a_0 \neq 0)$$

$$(-1)^{s+1} = 1 \Rightarrow s = 1$$

$$a_1 = (-1)^{s+1}(-a_1)$$

$$= (-1)^s a_1$$

$$a_1 = -a_1 \Rightarrow a_1 = 0$$

All a_ν with odd ν must be 0.

$$H(\xi) = \xi(a_0 + a_2\xi^2 + \dots)$$

$$= a_0\xi + a_2\xi^3 + \dots$$

In summary,

$$H(\xi) = \xi^s(a_0 + a_2\xi^2 + a_4\xi^4 + \dots),$$

where $s=0$ gives even H and

$s=1$ gives odd H.

$$u(\xi) = H(\xi)\, e^{-\frac{1}{2}\xi^2}$$

Question: Can H be an infinite series?

Let us assume (for a moment) that H is an infinite series,

$$\frac{a_{\nu+2}}{a_\nu} = \frac{2\nu+2s+1-\lambda}{(\nu+s+2)(\nu+s+1)}$$

$$\xrightarrow[\nu\to\infty]{} \frac{2\nu}{\nu^2} = \frac{2}{\nu}$$

In this case, $H(\xi)$ would behave as $\xi^n e^{2\xi^2}$.

$$\xi^n e^{2\xi^2} = \sum_{k=0}^{\infty} \frac{\xi^{2k+n}\, 2^k}{k!}$$

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\xi^n e^{2\xi^2} = \sum_{k=0}^{\infty} b_{2k+n}\, \xi^{2k+n}$$

$$b_{2k+n} = \frac{2^k}{k!}$$

$$\frac{b_{\overbrace{2k+n}^{\nu+2}}}{b_{\underbrace{2(k-1)+n}_{\nu}}} = \frac{b_{2k+n}}{b_{2k+n-2}}$$

$$= \frac{2^k\,(k-1)!}{k!\,2^{k-1}} = \frac{2}{k}$$

This implies that

$$H(\xi) \underset{|\xi|\to\infty}{\sim} \xi^n e^{2\xi^2}$$

or

$$u(\xi) \underset{|\xi|\to\infty}{\sim} \xi^n e^{2\xi^2} e^{-\frac{1}{2}\xi^2}$$

$$= \xi^n e^{\frac{3}{2}\xi^2}$$

$$\to \infty \quad \text{as } |\xi|\to\infty.$$

We must reject this and conclude that $H(\xi)$ is a finite series or a polynomial.

$$H(\xi) = \xi^s(a_0 + a_2\xi^2 + \dots + a_{2k}\xi^{2k}),$$ ✓

where $k \geq 0$ and $s = 0$ or 1.

H is a POLYNOMIAL of order $(2k+s)$ and

$$a_{2k+2} = a_{2k+4} = \dots = 0.$$

$$a_{\nu+2}(\nu+s+2)(\nu+s+1) - (2\nu+2s+1-\lambda)a_\nu = 0$$

Use $\nu = 2k$,

$$a_{2k+2}(2k+s+2)(2k+s+1) - (4k+2s+1-\lambda)a_{2k} = 0$$

$$a_{2k+2} = 0, \quad a_{2k} \neq 0$$

$$(4k+2s+1-\lambda)a_{2k} = 0$$

$$a_{2k} \neq 0$$

$$4k+2s+1-\lambda = 0$$

$$\boxed{\begin{aligned}\lambda &= 4k+2s+1 \quad \checkmark \\ &= 2(\underbrace{2k+s}_{\text{polynomial order}})+1\end{aligned}}$$

DEFINE

polynomial order

$$n = 2k + s$$

$$n = 2k+s = \begin{cases} 2k \text{ (even)}, & s=0 \\ 2k+1 \text{ (odd)}, & s=1 \end{cases}$$

$H(\xi)$ is a polynomial of order n that has a parity of n (H is even when n is even, H is odd when n is odd),

$$H(\xi) \equiv H_n(\xi),$$

$$H_n(-\xi) = (-1)^n H_n(\xi).$$

$$\lambda = \lambda_n = 2n+1\,, \quad n = 0, 1, \ldots$$

$$\lambda = \frac{2E}{\hbar\omega}$$

$$\lambda_n = \frac{2E_n}{\hbar\omega} = 2n+1$$

$$E = E_n = \hbar\omega\left(n+\frac{1}{2}\right),$$

$$n = 0, 1, \ldots$$

normalization

$$u = u_n = N_n H_n(\xi) e^{-\frac{1}{2}\xi^2}$$

$$H_n'' - 2\xi H_n' + 2n H_n = 0$$

H_n is a polynomial of order n and parity of n. (Hermite polynomials)