Theorem: Two observables F_1 and F_2 are simultaneously measurable, i.e. share a complete set of eigenstates if and only if $[\hat{F}_1, \hat{F}_2] = 0$.

Example:

$$u_{\vec{k}}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$

$$\hat{\vec{p}}\, u_{\vec{k}} = (\hbar \vec{k})\, u_{\vec{k}}$$

$u_{\vec{k}}$ are eigenstates of $\hat{T}$.

$$[\hat{\vec{p}}, \hat{T}] = [\hat{\vec{p}}, \frac{\hat{\vec{p}}^2}{2m}]$$

$$= 0.$$

$u_{\vec{k}}$'s are NOT eigenstates of $\hat{\vec{L}}^2$ and $\hat{L}_z$, because

$$[\hat{p}_x, \hat{L}_z] = [\hat{p}_x, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x]$$

$$\neq 0.$$

WARNING:

It may happen that $[\hat{F}_1, \hat{F}_2] \neq 0$ and yet $\hat{F}_1$ and $\hat{F}_2$ share SOME eigenstates (see homework # 6, problem 4).

THE HEISENBERG UNCERTAINTY PRINCIPLE

$$\Delta x \, \Delta p_x \gtrsim \hbar$$

or

$$\Delta E \, \Delta t \gtrsim \hbar$$

will be related to $[\hat{F}, \hat{G}]$

F, G,

$$(\Delta F)(\Delta G) \geqslant ?$$

Definition of ΔF (uncertainty of F):

$$\Delta \hat{F} = \hat{F} - \langle \varphi | \hat{F} | \varphi \rangle,$$

$$\|\varphi\| = 1.$$

$$\langle \hat{F} \rangle \equiv \langle \varphi | \hat{F} | \varphi \rangle$$

$$\Delta \hat{F} = \hat{F} - \langle \hat{F} \rangle$$

$\Delta \hat{F}$ is self-adjoint because $\hat{F}$ is self-adjoint and $\langle \hat{F} \rangle$ is a real number,

$$(\Delta \hat{F})^\dagger = \hat{F}^\dagger - \langle \hat{F} \rangle$$

$$= \hat{F} - \langle \hat{F} \rangle$$

$$= \Delta \hat{F} .$$

$$\Delta F = \langle (\Delta \hat{F})^2 \rangle^{\frac{1}{2}} \quad \checkmark$$

$$\Delta F = \langle (\hat{F} - \langle \hat{F} \rangle)^2 \rangle^{\frac{1}{2}}$$

$$\Delta F = \langle \varphi | (\hat{F} - \langle \varphi | \hat{F} | \varphi \rangle)^2 | \varphi \rangle^{\frac{1}{2}}$$

The uncertainty of $\hat{F}$ is the standard deviation for F. In statistics,

$$\sigma = \sqrt{\overline{(F - \overline{F})^2}}$$

$$= \sqrt{\overline{F^2} - (\overline{F})^2} \quad ✓$$

$$(\Delta F)^2 = \langle (\hat{F} - \langle \hat{F} \rangle)^2 \rangle$$

$$= \langle \hat{F}^2 - 2\langle \hat{F} \rangle \hat{F} + \langle \hat{F} \rangle^2 \rangle$$

$$(\Delta F)^2 = \langle \hat{F}^2 \rangle - \cancel{2}\langle \hat{F} \rangle^2 + \cancel{\langle \hat{F} \rangle^2} = \langle \hat{F}^2 \rangle - \langle \hat{F} \rangle^2 \quad \checkmark$$

$$\Delta F = \left(\langle \hat{F}^2 \rangle - \langle \hat{F} \rangle^2 \right)^{\frac{1}{2}}$$

$$\Delta F \geq 0$$

$$(\Delta F)^2 = \langle (\Delta \hat{F})^2 \rangle$$

$$= \langle \varphi | (\Delta \hat{F})(\Delta \hat{F}) | \varphi \rangle$$

$$\underset{\Delta \hat{F}^\dagger = \Delta \hat{F}}{\overset{\text{TOR}}{=}} \| (\Delta \hat{F}) \varphi \|^2 \geq 0$$

What if $\Delta F = 0$?

$$\|(\Delta\hat{F})\varphi\| = 0$$

$$\Delta\hat{F}|\varphi\rangle = \Theta$$

$$(\hat{F} - \langle\varphi|\hat{F}|\varphi\rangle)|\varphi\rangle = \Theta$$

$$\hat{F}|\varphi\rangle = \underbrace{\langle\varphi|\hat{F}|\varphi\rangle}_{\text{real number}}|\varphi\rangle$$

eigenstate   eigenvalue

PROOF OF THE HEISENBERG RELATION.

Let us consider $\hat{F}$ and $\hat{G}$ and define

$$\Delta\hat{F} = \hat{F} - \langle \hat{F} \rangle$$

$$\Delta\hat{G} = \hat{G} - \langle \hat{G} \rangle$$

We know that $(\Delta\hat{F})^{\dagger} = \Delta\hat{F}$ and $(\Delta\hat{G})^{\dagger} = \Delta\hat{G}$.

Let us introduce

arbitrary real number

$$\hat{A} = \Delta\hat{F} + i\alpha\Delta\hat{G}$$

$$I(\alpha) = \langle \varphi | \hat{A}^{\dagger}\hat{A} | \varphi \rangle \underset{\text{TDR}}{=} \| \hat{A}\varphi \|^2 \geq 0, \quad \forall_{\alpha \in \mathbb{R}}$$

$$I(\alpha) = \langle \varphi | (\Delta\hat{F} - i\alpha\Delta\hat{G})$$

$$\times (\Delta\hat{F} + i\alpha\Delta\hat{G}) | \varphi \rangle$$

$$= \langle \varphi | (\Delta\hat{F})^2 + i\alpha(\Delta\hat{F})(\Delta\hat{G})$$

$$- i\alpha(\Delta\hat{G})(\Delta\hat{F}) + \alpha^2(\Delta\hat{G})^2 | \varphi \rangle$$

$$I(\alpha) = (\Delta F)^2$$

$$+ i\alpha \langle (\Delta\hat{F})(\Delta\hat{G}) - (\Delta\hat{G})(\Delta\hat{F}) \rangle$$

$$+ \alpha^2 (\Delta G)^2$$

$$[\Delta\hat{F}, \Delta\hat{G}] = [\hat{F} - \langle \hat{F} \rangle,$$

$$\hat{G} - \langle \hat{G} \rangle] = [\hat{F}, \hat{G}]$$

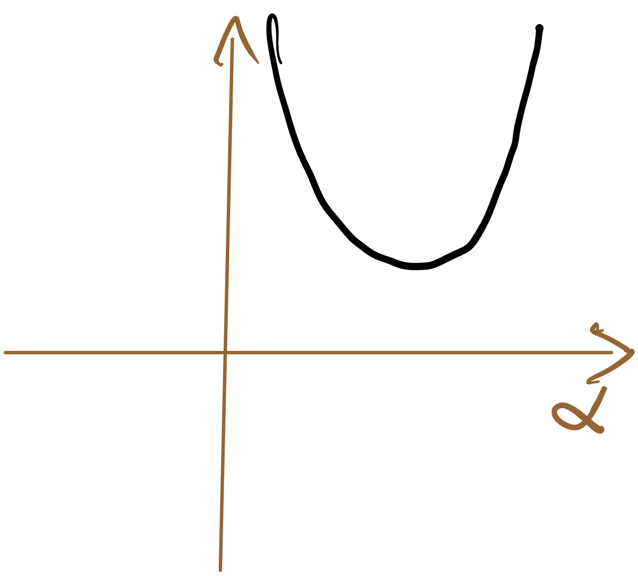
$$I(\alpha) = (\Delta F)^2 + i\alpha\langle[\hat{F},\hat{G}]\rangle + \alpha^2(\Delta G)^2$$

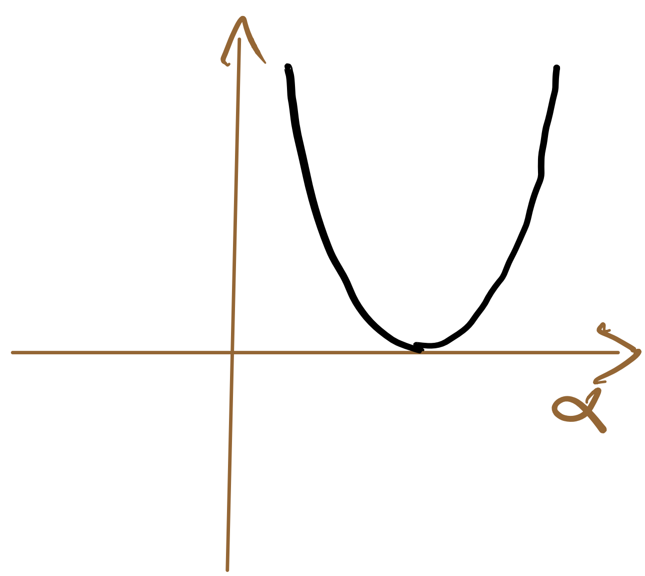
$$I(\alpha) = a\alpha^2 + b\alpha + c \geqslant 0$$

$$a = (\Delta G)^2 \geqslant 0 \quad \checkmark$$

$$b = i\langle[\hat{F},\hat{G}]\rangle$$

$$c = (\Delta F)^2$$





$$\Delta = b^2 - 4ac \leq 0$$

$$\Delta = -\langle [\hat{F},\hat{G}] \rangle^2 - 4(\Delta F)^2(\Delta G)^2 \leq 0$$

$$\boxed{(\Delta F)^2(\Delta G)^2 \geq -\frac{1}{4}\langle [\hat{F},\hat{G}] \rangle^2}$$

We will show now that

$$-\frac{1}{4}\langle [\hat{F},\hat{G}] \rangle^2 \geq 0$$

Proof:

$$\hat{B} = [\hat{F},\hat{G}]$$

$$\hat{B}^\dagger = [\hat{F},\hat{G}]^\dagger = (\hat{F}\hat{G} - \hat{G}\hat{F})^\dagger$$

$$= \hat{G}^\dagger \hat{F}^\dagger - \hat{F}^\dagger \hat{G}^\dagger$$

$$= \hat{G}\hat{F} - \hat{F}\hat{G} = -[\hat{F},\hat{G}]$$

$$= -\hat{B}$$

$$\hat{B}^{\dagger} = -\hat{B}$$

$$\langle [\hat{F},\hat{G}] \rangle = \langle \hat{B} \rangle$$

$$= (\varphi, \hat{B}\varphi) \overset{\text{TOR}}{=} (\hat{B}^{\dagger}\varphi, \varphi)$$

$$= -(\hat{B}\varphi, \varphi) \overset{(a)}{=} -(\varphi, B\varphi)^{*}$$

$$= -\langle \hat{B} \rangle^{*}$$

$$\underbrace{\langle [\hat{F},\hat{G}] \rangle}_{a+bi} = -\langle [\hat{F},\hat{G}\rangle^{*}$$

$$a+bi = -(a+bi)^{*}$$

$$a+\cancel{bi} = -a+\cancel{bi}$$

$a = -a \Rightarrow a = 0$

$\langle [\hat{F},\hat{G}] \rangle$ is purely imaginary

$\langle [\hat{F},\hat{G}] \rangle = \beta i, \; \beta \in \mathbb{R}$

$\langle [\hat{F},\hat{G}] \rangle^2 = -\beta^2 \leq 0$

This means that

$(\Delta F)^2 (\Delta G)^2 \geq -\frac{1}{4} \langle [\hat{F},\hat{G}] \rangle^2 \geq 0$

Consequences:

- If $[\hat{F},\hat{G}] = 0$, $(\Delta F)^2 (\Delta G)^2 \geq 0$

$$(\Delta F)(\Delta G) \geq 0.$$

Example,

$$\hat{F} = \hat{q}_k, \ \hat{G} = \hat{p}_\ell, k \neq \ell$$

$$[\hat{q}_k, \hat{p}_\ell] = 0$$

$$\Delta q_k \Delta p_\ell \geq 0, \ k \neq \ell$$

$$(\Delta x \Delta p_y \geq 0)$$

– Consider

$$\hat{F} = \hat{q}_\ell, \ \hat{G} = \hat{p}_\ell,$$

$$[\hat{F}, \hat{G}] = [\hat{q}_\ell, \hat{p}_\ell] = i\hbar \mathbb{1}$$

$$(\Delta q_\ell)^2 (\Delta p_\ell)^2 \geq -\frac{1}{4} \langle [\hat{q}_\ell, \hat{p}_\ell] \rangle^2$$

$$= -\frac{1}{4}\langle (i\hbar)^2\rangle = \frac{\hbar^2}{4}$$

$$\left(\langle \mathbb{1}\rangle = \langle \varphi|\mathbb{1}|\varphi\rangle = \|\varphi\|^2 = 1\right)$$

$$(\Delta q_l)^2(\Delta p_l)^2 \geq \frac{\hbar^2}{4}$$

$$\Delta q_l \Delta p_l \geq \frac{\hbar}{2}$$

$$\boxed{\Delta q_k \, \Delta p_l \geq \frac{\hbar}{2}\delta_{kl}}$$

examples:

$$\Delta x \Delta p_y \geq 0$$

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

$$\hat{q}_3 = \hat{\varphi}, \quad \hat{p}_3 = \hat{p}_\varphi = \hat{L}_z$$

$$\Delta\varphi \, \Delta L_z \geq \frac{\hbar}{2}$$

— Consider, symbolically,

only for this analysis

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \quad \text{(energy operator)}$$

$$\hat{t} : \hat{t} f = t f \quad \text{(time operator)}$$

$$(\Delta E)^2 (\Delta t)^2 \geq -\frac{1}{4} \langle [\hat{E}, \hat{t}] \rangle^2$$

$$[\hat{E}, \hat{t}] f = i\hbar \frac{\partial}{\partial t}(t f) - t \, i\hbar \frac{\partial f}{\partial t}$$

$$= i\hbar f + \cancel{i\hbar t \frac{\partial f}{\partial t}}$$

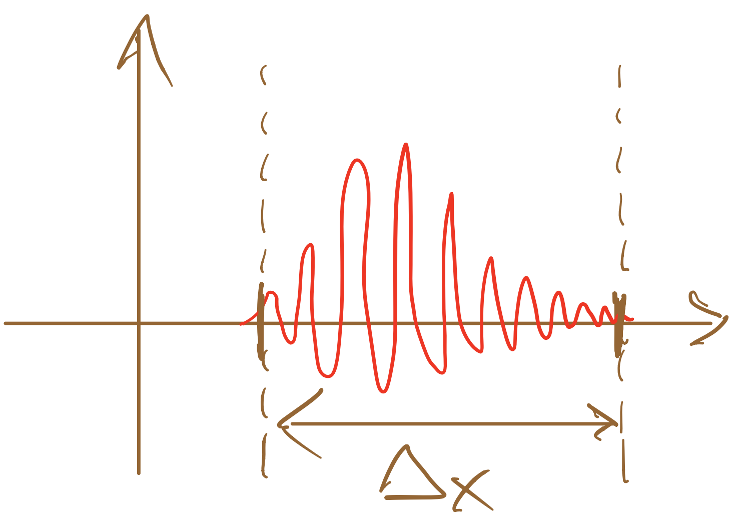
$$- \cancel{i\hbar t \frac{\partial f}{\partial t}} = i\hbar f$$

$$[\hat{E}, \hat{t}] = i\hbar \mathbb{1}$$

$$(\Delta E)^2 (\Delta t)^2 \geq -\frac{1}{4} \langle (i\hbar) \rangle^2$$

$$= \frac{\hbar^2}{4}$$

$$(\Delta E)(\Delta t) \geq \frac{\hbar}{2}$$



$$\hbar k = p_x$$

$$\downarrow$$

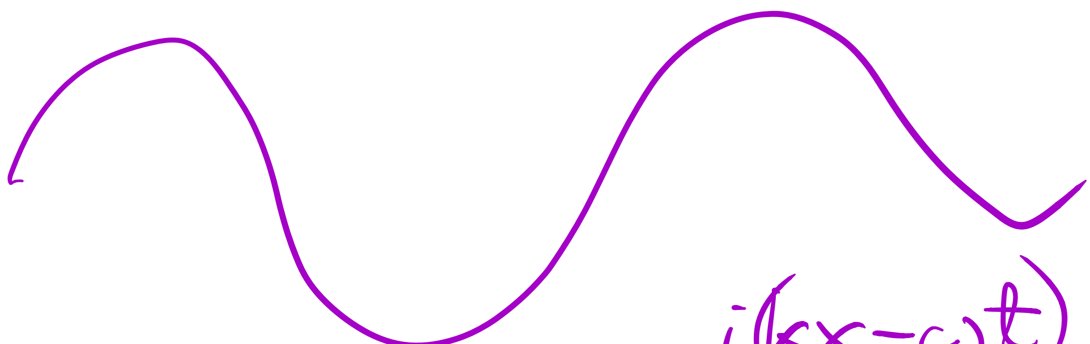
$$\Delta x \, \Delta k \geq \ldots$$

$$\Delta x \, \Delta p_x \geq \frac{\hbar}{2}$$

$$\hbar\omega = E$$

$$\Delta t \, \Delta\omega \geq \ldots$$

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$



$$\sim e^{i(kx-\omega t)}$$

$$\Delta k = 0, \quad \Delta p_x = 0$$

$$\Delta\omega = 0, \quad \Delta E = 0$$

$$\Delta x = \infty, \quad \Delta t = \infty$$

In reality,

$$\varphi \xrightarrow[\text{measurement}]{} \tilde{\varphi}$$

$$\langle \hat{F} \rangle_{\tilde{\varphi}} = \langle \tilde{\varphi} | \hat{F} | \tilde{\varphi} \rangle$$

$$(\Delta F)^2_{\tilde{\varphi}} (\Delta G)^2_{\tilde{\varphi}} \geqslant -\frac{1}{4} \langle [\hat{F}, \hat{G}] \rangle^2_{\tilde{\varphi}}$$

Even if $\tilde{\varphi}$ is very close to φ, we still have limitations; in the $\tilde{\varphi} \to \varphi$ limit,

$$(\Delta F)^2_{\varphi} (\Delta G)^2_{\varphi} \geqslant -\frac{1}{4} \langle [\hat{F}, \hat{G}] \rangle^2_{\varphi}$$

MINIMUM WAVE PACKET,

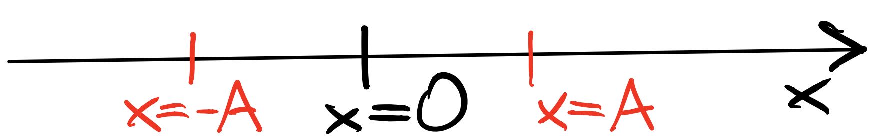
$$(\Delta F)^2 (\Delta G)^2 = -\frac{1}{4} \langle [\hat{F}, \hat{G}] \rangle^2$$

The ground state of a harmonic oscillator is a minimum wave packet for $\hat{F} = \hat{x}$, $\hat{G} = \hat{p}_x$,

$$(\Delta x)(\Delta p_x) = \frac{\hbar}{2}.$$

HARMONIC OSCILLATOR (in 1D)

In classical mechanics,



$$x(t) = A\sin(\omega t + \varphi) \quad \checkmark$$

$$F_x = -kx \quad \text{(Hooke's law)}$$

$$V(x) = \frac{kx^2}{2}$$

$$k = m\omega^2$$

angular frequency $\left(\omega = 2\pi\nu = \frac{2\pi}{T}\right)$

force constant

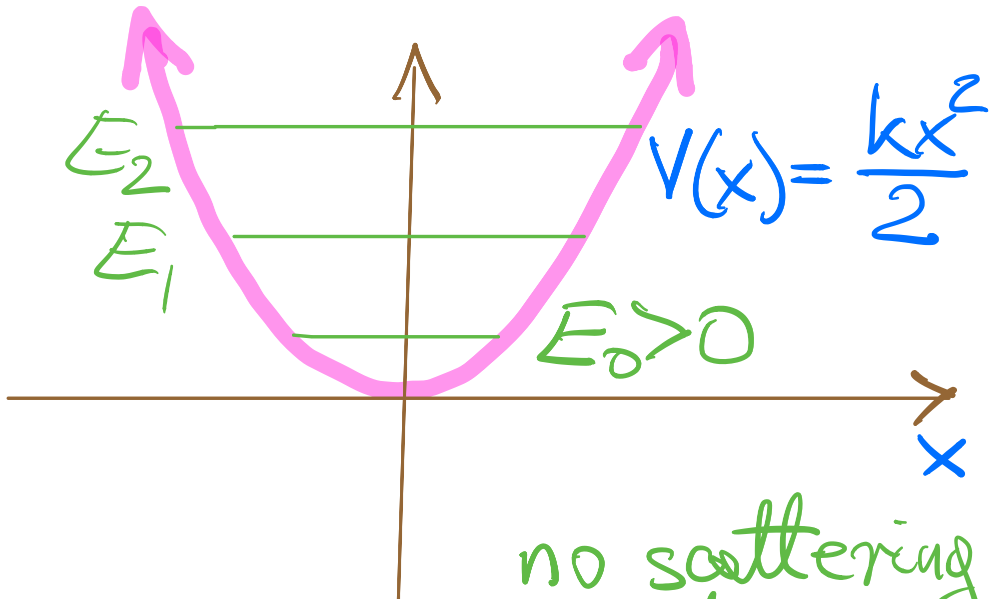
In QM,

$$\hat{H}u = E\hat{u}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{kx^2}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2}+\frac{1}{2}kx^2u=Eu$$

$$u=u(x)$$



no scattering states
(only discrete spectrum, only bound states)

In dimensionless form,

$$\frac{d^2u}{d\xi^2} + (\lambda - \xi^2)u = 0$$

$$\xi = \alpha x$$

What is α, what is λ?

$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \alpha\frac{du}{d\xi}$$

$$\frac{d^2u}{dx^2} = \alpha^2\frac{d^2u}{d\xi^2} \quad \checkmark$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + \frac{1}{2}kx^2u = Eu$$

$$/ \times\left(-\frac{2m}{\hbar^2}\right)$$

$$\frac{d^2u}{dx^2} - \frac{1}{\cancel{2}}\,\frac{\cancel{2}mk}{\hbar^2}\,x^2u = -\frac{2mE}{\hbar^2}\,u$$

$$\times\frac{1}{\alpha^2}\Big/\ \frac{d^2u}{dx^2} - \frac{mk}{\hbar^2}\,x^2u + \frac{2mE}{\hbar^2}\,u = 0$$

$$\frac{1}{\alpha^2}\,\frac{d^2u}{dx^2} - \overbrace{\frac{mk}{\hbar^2\alpha^4}}^{=1}\,\underbrace{\alpha^2x^2}_{\xi^2}\,u + \underbrace{\frac{2mE}{\hbar^2\alpha^2}}_{\to\,\lambda}\,u = 0$$

$$\frac{d^2u}{d\xi^2} + (\lambda - \xi^2)\,u = 0$$

$$\frac{mk}{\hbar^2\alpha^4} = 1$$

$$\lambda = \frac{2mE}{\hbar^2\alpha^2}$$

$$\alpha^4 = \frac{mk}{\hbar^2} = \frac{m\overbrace{(m\omega^2)}^{k}}{\hbar^2}$$

$$\alpha^4 = \frac{m^2\omega^2}{\hbar^2}$$

$$\alpha^2 = \frac{m\omega}{\hbar}$$

$$\lambda = \frac{2\cancel{m}E}{\hbar^{\cancel{2}}\,\frac{\cancel{m}\omega}{\cancel{\hbar}}} = \frac{2E}{\hbar\omega}$$

$$\frac{d^2u}{d\xi^2} + (\lambda - \xi^2)u = 0$$

$$\xi = \alpha x$$

$$\alpha = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}$$

$$\lambda = \frac{2E}{\hbar\omega}$$