$$G \subseteq \mathcal{H}$$

$$P_G h \in G$$ (P_G projects h on G)

P_G is linear, idempotent, and hermitian (self-adjoint)

$$h = g + f$$

$$g = P_G h \in G$$

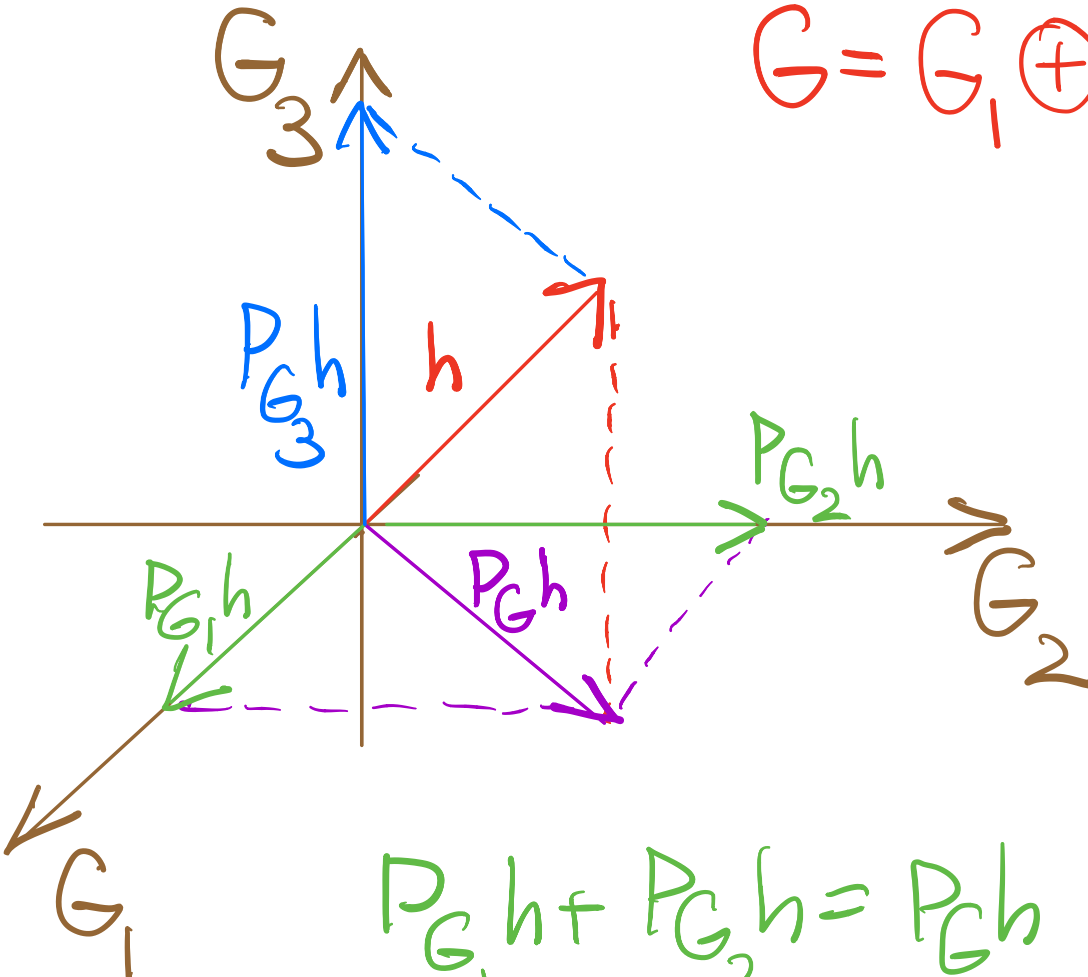
$$f = P_{G^\perp} h \in G^\perp$$

$$G = G_1 \oplus \dots \oplus G_n \checkmark$$

$$g = g_1 + \dots + g_n, \quad g_i \in G_i$$

$$P_G = P_{G_1} + \dots + P_{G_n} \checkmark$$

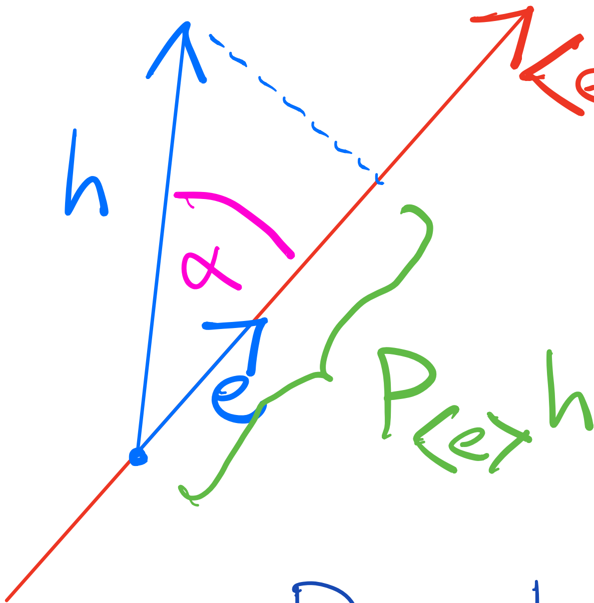
$G = G_1 \oplus G_2$



$$P_{G_1}h + P_{G_2}h = P_G h$$

$$h = P_{G_1}h + P_{G_2}h + P_{G_3}h$$

$$P_{G_1} + P_{G_2} = P_G$$



$\langle e\rangle = \mathrm{Span}\{e\}$

$(\lambda e, \lambda \in \mathbb{C})$

$\|e\| = 1$

$$P_{\langle e\rangle}h = (e,h)e$$

In Dirac's notation

$$P_{\langle e\rangle} = |e\rangle\langle e|$$

Example:

$\mathcal{H}$, $\dim \mathcal{H} = n$

$e_1, \dots, e_n \rightarrow$ orthonormal basis vectors

$\mathcal{H} = \operatorname{Span}\{e_1, \dots, e_n\}$

$$h = \underset{\langle e_1 \rangle}{\underset{\cap}{\lambda_1 e_1}} + \dots + \underset{\langle e_n \rangle}{\underset{\cap}{\lambda_n e_n}}$$

$$\mathcal{H} = \langle e_1 \rangle \oplus \dots \oplus \langle e_n \rangle$$

$$= \bigoplus_{i=1}^{n} \langle e_i \rangle \quad \checkmark$$

$$\mathbb{1} = P_{\mathcal{H}} = P_{\langle e_1 \rangle} + \ldots + P_{\langle e_n \rangle}$$

$$(P_{\mathcal{H}} h = h)$$

$$\mathbb{1} = \sum_{i=1}^{n} |e_i\rangle\langle e_i|$$

COMPLETE ORTHONORMAL SYSTEMS, BASIS

$$e_1, \ldots, e_n, \ldots = \{e_i\}_{i=1}^{\infty}$$

$$(e_i, e_j) = \delta_{ij}, \quad \|e_i\| = 1$$

$$G = \mathrm{Span}\{e_1, \dots, e_n, \dots\}$$

$$= \mathrm{Span}\{e_i\}_{i=1}^{\infty}$$

✓ $\sum_i \lambda_i e_{k_i}, \ 1 \leq k_i < \infty$

$\overline{G}$ = G plus all limits of sequences of vectors from G ✓

$$\overline{G} = \langle e_1, \dots, e_n, \dots \rangle \checkmark$$

$$G = \langle e_1 \rangle \oplus \dots \oplus \langle e_n \rangle \oplus \dots$$

$$G = \bigoplus_{i=1}^{\infty} \langle e_i \rangle \quad W$$

Vectors $e_1, \dots, e_n, \dots$ are linearly independent.

Proof:

$$(e_k, \; / \quad \sum_{i=1}^{n} \lambda_i e_i = \overset{\text{zero vector}}{\Theta}$$

$$\sum_{i=1}^{n} \lambda_i \underbrace{(e_k, e_i)}_{\delta_{ki}} = 0$$

$$\lambda_k = 0$$

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0 \; \therefore$$

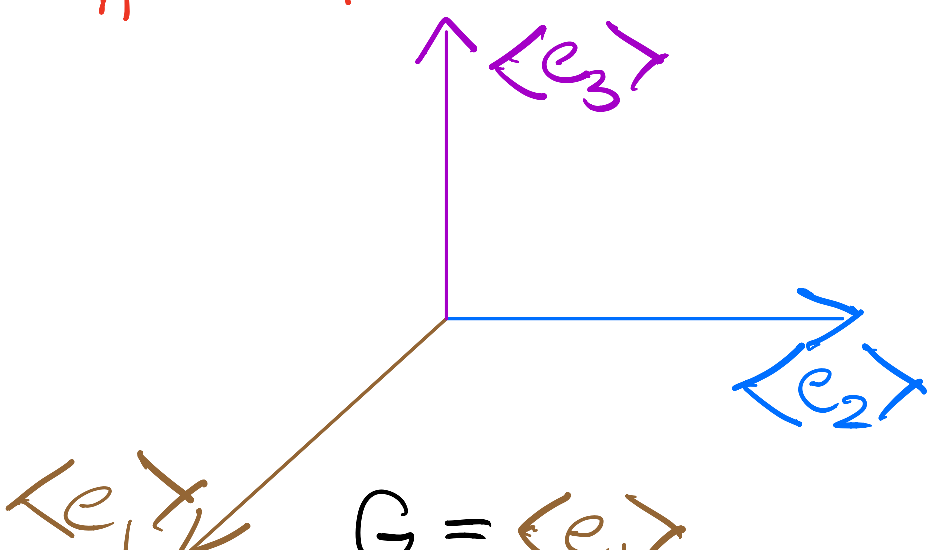
$$G_1 = \langle e_1 \rangle$$

$$G_2 = \langle e_1 \rangle \oplus \langle e_2 \rangle$$

$$G_3 = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle$$

$$\cdots$$

$$G_n = \langle e_1 \rangle \oplus \cdots \oplus \langle e_n \rangle$$



$$G_1 = \langle e_1 \rangle$$

$$G_2 = \langle e_1 \rangle \oplus \langle e_2 \rangle$$

$$G_3 = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle$$

$$P_{G_n} = \sum_{i=1}^{n} P_{\langle e_i \rangle}$$

$$P_{G_n} h = \sum_{i=1}^{n} P_{\langle e_i \rangle} h$$

$$= \sum_{i=1}^{n} (e_i, h) e_i \checkmark$$

$$h = P_{G_n} h + f_n$$

$$f_n \in G_n^{\perp}$$

We must learn how to calculate $\|h - P_{G_n} h\|$.

$$\delta_n = \| h - P_{G_n} h \| \quad \checkmark$$

$$\delta_n^2 = (h - P_{G_n} h, h - P_{G_n} h)$$

$$= (h,h) - (\overset{\text{TOR}}{P_{G_n} h, h})$$

$$- (h, P_{G_n} h) + (\overset{\text{TOR}}{P_{G_n} h, P_{G_n} h})$$

$$= \|h\|^2 - (h, P_{G_n} h)$$

$$- (h, P_{G_n} h) + (h, \overset{= P_{G_n}}{P_{G_n}^2} h)$$

$$= \|h\|^2 - \cancel{2}(h, P_{G_n} h)$$

$$+ \cancel{(h, P_{G_n} h)}$$

$$\delta_n^2 = \|h - P_{G_n} h\|^2$$
$$= \|h\|^2 - (h, P_{G_n} h)$$

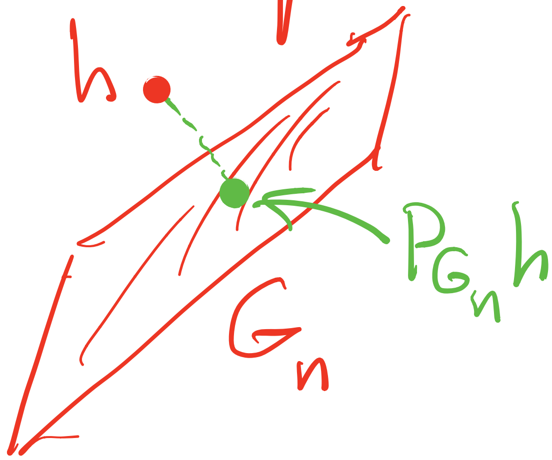
$$P_{G_n} h = \sum_{i=1}^{n} (e_i, h) e_i$$

$$\delta_n^2 = \|h\|^2 - \sum_{i=1}^{n} \underbrace{(e_i, h)}_{(h, e_i)^*} (h, e_i)$$

$$\delta_n^2 = \|h\|^2 - \sum_{i=1}^{n} |(h, e_i)|^2$$

Distance between h and $P_{G_n} h$ (between h and G_n) is

$$\delta_n = \sqrt{\|h\|^2 - \sum_{i=1}^{n} |(h,e_i)|^2}$$



$$\delta_n \geq 0, \quad \delta_n^2 \geq 0$$

$$\|h\|^2 - \sum_{i=1}^{n} |(h,e_i)|^2 \geq 0$$

$$\underbrace{\sum_{i=1}^{n} |(h,e_i)|^2}_{\text{bounded, so it converges as } n \to \infty} \leq \|h\|^2$$

$\lim_{n\to\infty} \sum_{i=1}^{n} |(h,e_i)|^2$ exists!

Let us calculate the distance between h and G (or between h and $P_G h$), $G = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$

$$\delta = \|h - P_G h\|$$
$$= \lim_{n\to\infty} \|h - P_{G_n} h\| \quad \checkmark$$

$$\delta^2 = \|h - P_G h\|^2$$
$$= \lim_{n\to\infty} \underbrace{\|h - P_{G_n} h\|^2}_{\delta_n^2}$$
$$= \|h\|^2 - \lim_{n\to\infty} \sum_{i=1}^{n} |(h, e_i)|^2$$

$$\delta^2 = \|h\|^2 - \sum_{i=1}^{\infty} |(h, e_i)|^2$$

Theorem:

The distance δ between $h \in \mathcal{H}$ and $G = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$ is

$$\delta = \sqrt{\|h\|^2 - \sum_{i=1}^{\infty} |(h, e_i)|^2}$$

The same formula applies to the distance between h and $\overline{G}$. ✓

closure of G

Suppose G is dense in $\mathcal{H}$,

$\overline{G} = \mathcal{H}$, i.e.,

$$\mathcal{H} = \langle e_1, \dots, e_n, \dots \rangle.$$

This means that

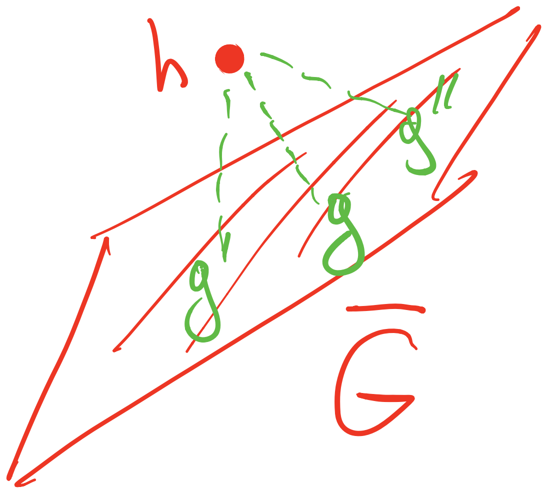
$$\forall_{h \in \mathcal{H}} \; \exists_{g_n \in G} \; g_n \xrightarrow{n \to \infty} h,$$

sequence

or $\|h - g_n\| \xrightarrow{n \to \infty} 0$.

$$\delta = \inf_{g' \in G} \|h - g'\| \leq \|h - g_n\| \xrightarrow{n \to \infty} 0$$

distance between h and $\overline{G}$



If $\overline{G} = \mathcal{H}$ then $\delta = 0$.

CLOSURE RELATION

Theorem (the Parseval relation)

If $\overline{G} = \mathcal{H}$, where

$$G = \bigoplus_{i=1}^{\infty} \langle e_i \rangle,$$

$$\|h\|^2 = \sum_{i=1}^{\infty} |(h, e_i)|^2.$$

Proof: Use $\delta = 0$ and the previous theorem.

Definition:

An orthonormal system of vectors $e_1, e_2, \dots$ is closed if $G = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$ is dense in $\mathcal{H}$.

For closed systems,

$$\forall_{h \in \mathcal{H}} \quad \|h\|^2 = \sum_{i=1}^{\infty} |(h, e_i)|^2$$

(the Parseval relation, the closure relation)

Definition:
An orthonormal system $\{e_1, \dots, e_n, \dots\}$ is complete in $\mathcal{H}$ if there exists no nonzero vector from $\mathcal{H}$ orthogonal to all vectors e_i, $i = 1, \dots$.

Theorem:
An infinite orthonormal system $\{e_1, \dots, e_n, \dots\}$ is complete iff it is closed in $\mathcal{H}$.

COMPLETE $\Longleftrightarrow$ CLOSED

only one way: closed $\Downarrow$ complete

Proof: Let us assume that $\{e_i\}_{i=1}^{\infty}$ is closed,

$$\forall_{h \in \mathcal{H}} \quad \|h\|^2 = \sum_{i=1}^{\infty} |(h, e_i)|^2 .$$

Suppose there exists $h \in \mathcal{H}$ which is orthogonal to all e_i's,

$$\|h\|^2 = \sum_{i=1}^{\infty} |\underbrace{(h, e_i)}_{=0}|^2 = 0$$

$$\|h\| = 0 \Rightarrow h = \Theta .$$

This means that $\{e_i\}_{i=1}^{\infty}$ is complete. $\therefore$

One can prove that two complete orthonormal systems in a given $\mathcal{H}$ have the same dimension (cardinal number).

An orthonormal BASIS = a complete orthonormal system! (which also means a closed system)

We have learned that basis $\{e_i\}_{i=1}^{\infty}$ has the following properties:

1. All e_i's are linearly independent.

2. $$\forall_{h \in \mathcal{H}} \left\| h - \sum_{i=1}^{n} (e_i, h) e_i \right\| \xrightarrow[n\to\infty]{} 0$$

$P_{G_n} h$

Symbolically,

$$h = \sum_{i=1}^{\infty} (e_i, h) e_i$$

In practice,

$$h = \sum_{i=1}^{\infty} c_i e_i$$

$$c_i = (e_i, h)$$

✓

$(e_k,\ /$

$$h=\sum_{i=1}^{\infty} c_i e_i$$

$$(e_k,h)=\sum_{i=1}^{\infty} c_i \underbrace{(e_k,e_i)}_{\delta_{ki}}$$

$$=c_k$$

$$c_k=(e_k,h). \quad \checkmark$$

Generalization of the Parseval relation:

$$\|h\|^2=(h,h)=\sum_{i=1}^{\infty}|(h,e_i)|^2$$

$$=\sum_{i=1}^{\infty}(h,e_i)(e_i,h)$$

$$\boxed{(g,h) = \sum_{i=1}^{\infty} (g,e_i)(e_i,h)}$$

Proof:

$(g+\lambda h, g+\lambda h)$ $\lambda \in \mathbb{C}$

$$= (g,g) + \lambda^*(h,g) + \lambda(g,h) + \lambda^*\lambda(h,h)$$

$$= \|g\|^2 + \lambda^*(h,g) + \lambda(g,h) + |\lambda|^2 \|h\|^2$$

$$(g+\lambda h, g+\lambda h) \overset{\text{Parseval}}{=} \sum_{i=1}^{\infty} (g+\lambda h, e_i)(e_i, g+\lambda h)$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} (g,e_i)(e_i,g) \xrightarrow{\text{Parseval}} \|g\|^2 \\
&+ \sum_{i=1}^{\infty} \lambda^*(h,e_i)(e_i,g) \\
&+ \sum_{i=1}^{\infty} \lambda(g,e_i)(e_i,h) \\
&+ \underbrace{\sum_{i=1}^{\infty} \lambda^*\lambda(h,e_i)(e_i,h)}_{|\lambda|^2\,\|h\|^2}
\end{aligned}
$$

$$
\overset{1}{\lambda^*(h,g)} + \overset{2}{\lambda(g,h)}
$$

$$
\begin{aligned}
&= \lambda^* \sum_{i=1}^{\infty} (h,e_i)(e_i,g) \quad 3 \\
&+ \lambda \sum_{i=1}^{\infty} (g,e_i)(e_i,h) \quad 4
\end{aligned}
$$

1 = 3

2 = 4

$$\cancel{\lambda}(g,h) = \cancel{\lambda}\sum_{i=1}^{\infty}(g,e_i)(e_i,h)$$

$\therefore\ \checkmark$

In QM,

$$(\varphi,\psi) = \langle\varphi|\psi\rangle$$

$\{u_i\}_{i=1}^{\infty} \rightarrow$ orthonormal basis in $\mathcal{H} = L^2(\Gamma)$

$$|\varphi\rangle = \sum_{i=1}^{\infty} c_i |u_i\rangle$$

$$c_i = (u_i,\varphi) = \langle u_i|\varphi\rangle$$

$$\langle \varphi | \psi \rangle = \sum_{i=1}^{\infty} \langle \varphi | u_i \rangle \langle u_i | \psi \rangle$$

$$\sum_{i=1}^{\infty} | u_i \rangle \langle u_i | = \mathbb{1} \quad \checkmark$$