DIRAC NOTATION

$(f, g) \rightarrow$ scalar product

$$(\varphi, \psi) = \int_{\Gamma} \varphi^* \psi \, d\tau$$

$$\underbrace{\langle \varphi | \psi \rangle}_{\text{bracket}} = (\varphi, \psi)$$

$\langle \varphi | =$ "bra" ✓

$| \psi \rangle =$ "ket" ✓

$$\underbrace{\langle \varphi |}_{\text{bra}} \cdot \underbrace{| \psi \rangle}_{\text{ket}} = \underbrace{\langle \varphi | \psi \rangle}_{\text{bracket}}$$

$|\psi\rangle$ - ket, $\psi \in \mathcal{H} = L^2(\Gamma)$

$\langle\varphi| = |\varphi\rangle^+$ ✓

A, $\underline{\underline{A}} = |A_{ij}|_{i,j=1,\dots,n}$

$\{e_1, \dots, e_n\}$ - orthonormal basis

$(e_i, e_j) = \delta_{ij}$

$A_{ij} = (e_i, Ae_j)$ ✓

$(e_i,)\!\!\!/\ \ A e_j = \sum_{k=1}^{n} A_{kj} e_k$

$(e_i, Ae_j) = \sum_{k=1}^{n} A_{kj} \underbrace{(e_i, e_k)}_{\delta_{ik}}$

$$(e_i, Ae_j) = A_{ij} \checkmark$$

$$(A^\dagger)_{ij} = (e_i, A^\dagger e_j)$$

$$\overset{(a)}{=} (A^\dagger e_j, e_i)^*$$

$$\overset{\text{TOR}}{=} (e_j, Ae_i)^* = A_{ji}^*$$

$$(A^\dagger)_{ij} = A_{ji}^* \checkmark$$

$$\underline{\underline{A^\dagger}} = (\underline{\underline{A^T}})^* \checkmark$$

$$|\varphi\rangle = \begin{bmatrix} \vdots \\ \varphi(x) \\ \vdots \end{bmatrix} \leftarrow \text{xth component of } |\varphi\rangle$$

$$|\psi\rangle = \begin{bmatrix} \vdots \\ \psi(x) \\ \vdots \end{bmatrix} \leftarrow \text{xth component of } |\psi\rangle$$

$$|\varphi\rangle^{\dagger} = [\cdots \ \varphi(x)^{*} \cdots]$$

$$\langle\varphi|\psi\rangle = (\varphi, \psi) = \int \varphi^{*} \psi \, dx$$

$$= \text{“}\sum_x\text{”}\, \varphi^*(x)\, \psi(x)$$

$$= \underbrace{\begin{bmatrix} \cdots \varphi(x)^* \cdots \end{bmatrix}}_{\langle\varphi|} \bullet \underbrace{\begin{bmatrix} \vdots \\ \psi(x) \\ \vdots \end{bmatrix}}_{|\psi\rangle}$$

$$|\varphi\rangle^\dagger = \langle\varphi| \quad \checkmark$$

$\mathbb{C}^n$:

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\underline{x}^\dagger = \begin{bmatrix} x_1^* \cdots x_n^* \end{bmatrix}$$

$$\underline{x}^{\dagger} \cdot \underline{y} = x_1^* y_1 + \cdots + x_n^* y_n$$
$$= (x, y)$$

$$|c_1\varphi_1 + c_2\varphi_2\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle$$

$$\langle c_1\varphi_1 + c_2\varphi_2| = c_1^*\langle\varphi_1| + c_2^*\langle\varphi_2|$$

Proof:

$$\langle c_1\varphi_1 + c_2\varphi_2| = |c_1\varphi_1 + c_2\varphi_2\rangle^{\dagger}$$
$$= \left(c_1|\varphi_1\rangle + c_2|\varphi_2\rangle\right)^{\dagger}$$
$$= c_1^*\,|\varphi_1\rangle^{\dagger} + c_2^*\,|\varphi_2\rangle^{\dagger}$$
$$= c_1^*\langle\varphi_1| + c_2^*\langle\varphi_2| \quad \therefore$$

$$(\varphi, \hat{F}\psi) = \int_{\Gamma} \varphi^* \hat{F}\psi \, dx$$

$$(\varphi, \hat{F}\psi) = \langle \varphi | \hat{F}\psi \rangle$$

$$= \langle \varphi | \hat{F} | \psi \rangle$$

$$F_{\varphi\psi}$$

$$\langle \hat{F} \rangle_{\varphi} = (\varphi, \hat{F}\varphi)$$

$$\|\varphi\| = 1$$

$$\langle \hat{F} \rangle_{\varphi} = \langle \varphi | \hat{F} | \varphi \rangle$$

$$(\varphi, \hat{F}\psi) \overset{\text{TOR}}{=} (\hat{F}^{\dagger}\varphi, \psi)$$

$$\langle \varphi | \hat{F}\psi \rangle = \langle \hat{F}^{\dagger}\varphi | \psi \rangle$$

$$\langle \varphi | \hat{F} | \psi \rangle = \langle \psi | \hat{F}^{\dagger}\varphi \rangle^{*}$$

$$\langle \varphi | \hat{F} | \psi \rangle = \langle \psi | \hat{F}^{\dagger} | \varphi \rangle^{*}$$

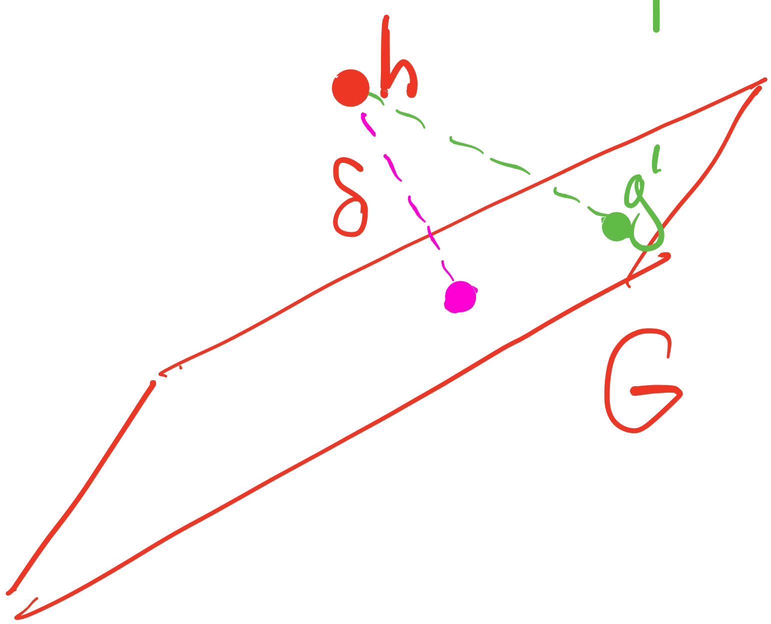
$$F_{\varphi\psi} \qquad (F^{\dagger}_{\psi\varphi})^{*}$$

What if $\hat{F} = \hat{F}^{\dagger}$?

$$\langle \varphi | \hat{F} | \psi \rangle = \langle \psi | \hat{F} | \varphi \rangle^{*}$$

PROJECTION OPERATORS

G – subspace of $\mathcal{H}$
(linear, complete)

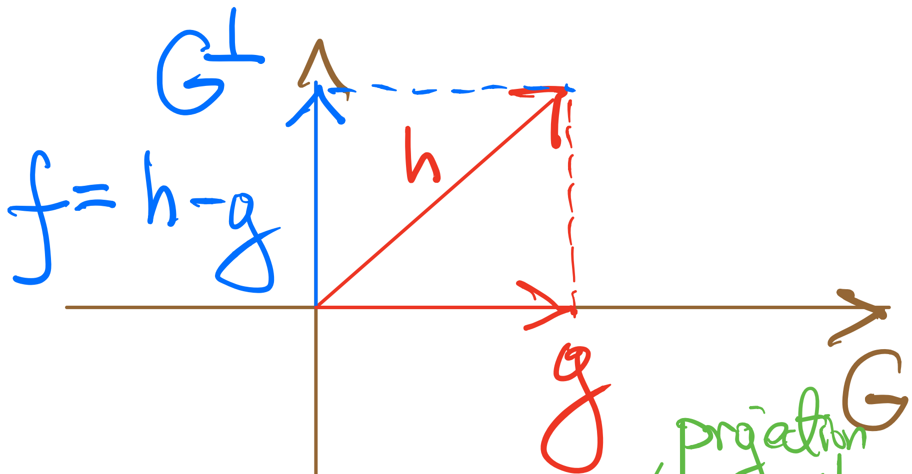


$h \in \mathcal{H}$

g is a vector from G closest to h.

$$\delta = \inf_{g' \in G} \|h - g'\|$$

such g exists and there is only one; see p.257



projection operator

$$g = P_G h \quad \longleftarrow \text{projection of } h \text{ on } G$$

$$\|h-g\| = \inf_{g'\in G} \|h-g'\|$$

$$h-g = h - P_G h \perp G$$

Proof: By contradiction.
Suppose $h-g$ is NOT orthogonal to G.

There must exist $g_0 \in G$ such that

$$\sigma = (g_0, h-g) \neq 0. \checkmark$$

Let us define

$$g_1 = \underset{\underset{G}{\cap}}{g} + \frac{\sigma}{\|g_0\|^2} \underset{\underset{G}{\cap}}{g_0} \in G$$

$$\|h-g_1\|^2 = (h-g_1, h-g_1)$$

$$= \Big(\underset{①}{(h-g)} - \underset{②}{\frac{\sigma}{\|g_0\|^2} g_0}, \underset{③}{(h-g)} - \underset{④}{\frac{\sigma}{\|g_0\|^2} g_0} \Big)$$

$$= (h-g, h-g) - \frac{\overbrace{(h-g, g_0)}^{\sigma^*}\,\sigma}{\|g_0\|^2}$$

$$\quad\;\; ①,③ \qquad\qquad\qquad ①,④$$

$$- \frac{\sigma^* \overbrace{(g_0, h-g)}^{\sigma}}{\|g_0\|^2} + \frac{\sigma^* \sigma (g_0, g_0)}{\|g_0\|^4}$$

$$\quad\;\; ②,③ \qquad\qquad\qquad ②,④$$

$$= \|h-g\|^2 - \frac{|\sigma|^2}{\|g_0\|^2}$$

$$\cancel{- \frac{|\sigma|^2}{\|g_0\|^2}} + \cancel{\frac{|\sigma|^2 \cancel{\|g_0\|^2}}{\|g_0\|^{\cancel{4}\,2}}}$$

$$\|h-g_1\|^2 = \|h-g\|^2 - \frac{|\sigma|^2}{\|g\|^2} < \|h-g\|^2$$

closest to h

$$\|h-g_1\| < \|h-g\|$$

NOT POSSIBLE

This means that

$$h-g = h-P_G h \perp G$$

$$\therefore$$

$$h = \underset{\substack{\cap \\ G}}{g} + \underset{\substack{\cap \\ G^{\perp}}}{(h-g)}$$

$$h = g + f$$

$$g = P_G h \in G$$

$$f \in \underset{\substack{\shortparallel \\ F}}{G^{\perp}} \leftarrow \text{is a subspace (see p. 260)}$$

$G^{\perp}$ – orthogonal complement to G

$$G^{\perp} = \mathcal{H} \ominus G \checkmark$$

$$h = g + f$$

$$g = P_G h \in G$$

$$f \in G^{\perp}$$

THIS DECOMPOSITION IS UNIQUE!

Proof by contradiction:

$$h = \underset{\substack{\cap \\ G}}{g} + \underset{\substack{\cap \\ G^{\perp}}}{f} = \underset{\substack{\cap \\ G}}{g'} + \underset{\substack{\cap \\ G^{\perp}}}{f'}$$

$$(g, f) = 0, \quad (g', f') = 0$$

$$\underbrace{g - g'}_{\substack{\cap \\ G}} = \underbrace{f' - f}_{\substack{\cap \\ G^{\perp}}} \qquad \checkmark$$

The only vector that belongs to G and $G^{\perp}$ is a zero vector Θ.

Suppose we can find $\tilde{h} \in G \cap G^{\perp}$,

$$0 = (\underset{\substack{\cap \\ G}}{\tilde{h}}, \underset{\substack{\cap \\ G^\perp}}{\tilde{h}}) = \|\tilde{h}\|^2 \Rightarrow \tilde{h} = \Theta$$

$$g - g' = f' - f = \Theta$$

$$g = g', \ f = f' \ \therefore$$

$$\mathcal{H} = G \oplus G^\perp \ \checkmark$$

$$h = \underset{\substack{\cap \\ G}}{g} + \underset{\substack{\cap \\ G^\perp}}{f}$$

$\mathcal{H}$ is an orthogonal sum of G and $G^\perp$.

$$g = P_G h$$

$$f = P_{G^\perp} h$$

$$P_{\mathcal{H}} h = h$$

$$P_{\mathcal{H}} h = h = g + f = P_G h + P_{G^\perp} h$$

$$P_{\mathcal{H}} h = P_G h + P_{G^\perp} h$$

$$P_{\mathcal{H}} = P_G + P_{G^\perp} \quad \forall_{h \in \mathcal{H}}$$

orthogonal sum of subspaces

$$G = G_1 \oplus \cdots \oplus G_n$$

$$G_i \perp G_j, \quad i \neq j$$

$$(g_i, g_j) = 0, \quad g_i \in G_i, \; g_j \in G_j, \; i \neq j$$

$$P_G = P_{G_1} + \dots + P_{G_n}$$

Proof: $g \in G$,

$\checkmark$ $g = g_1 + \dots + g_n, \quad g_i \in G_i$

$$g_i = P_{G_i} g$$

Suppose we consider

$$h \in \mathcal{H}, \quad g = P_G h$$

$$h = g + f, \quad g \in G$$
$$f \in G^{\perp}$$

$$h = g_1 + \cdots + g_n + f$$

$$g_1 \in G_1, \ldots, g_n \in G_n,$$
$$f \in G^{\perp}, \quad G^{\perp} \perp G_i$$
$$(i = 1, \ldots, n)$$

$$P_{G_i} h = g_i$$

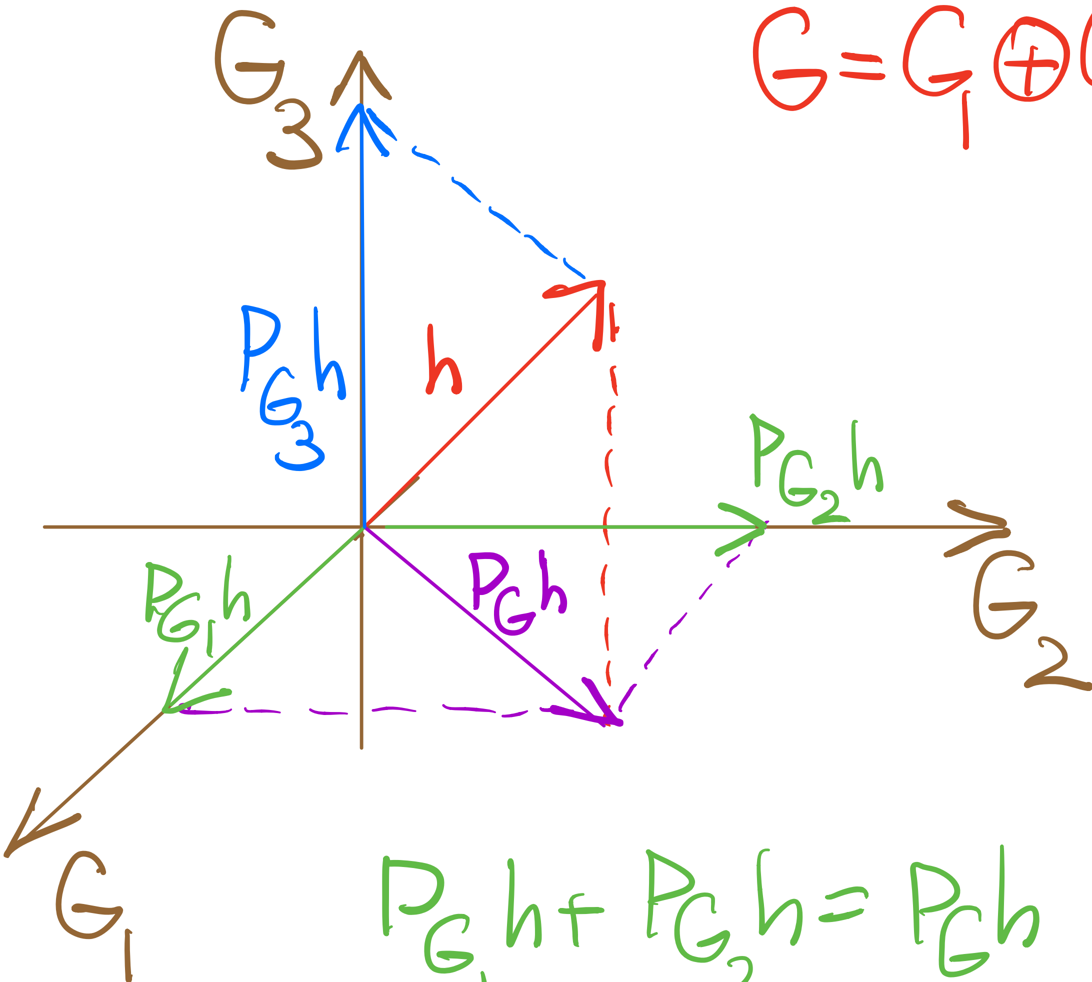
$$P_G h = g = g_1 + \cdots + g_n$$

$$= P_{G_1} h + \cdots + P_{G_n} h$$

$$= (P_{G_1} + \cdots + P_{G_n}) h$$

$$\boxed{\begin{array}{l} P_G = P_{G_1} + \cdots + P_{G_n} \\ \text{if} \\ G = G_1 \oplus \cdots \oplus G_n \end{array}}$$

$$G = G_1 \oplus G_2$$



$$P_{G_1}h + P_{G_2}h = P_G h \quad \checkmark$$

$$h = P_{G_1}h + P_{G_2}h + P_{G_3}h$$

$$P_{G_1} + P_{G_2} = P_G \quad \checkmark$$

$$G = G_1 \oplus G_2$$

Projection operators are linear, idempotent, and hermitian.

Linearity:

$$h = \alpha_1 h_1 + \alpha_2 h$$

$$h_1 = g_1 + f_1, \quad g_1 = P_G h_1 \in G$$

$$h_2 = g_2 + f_2, \quad g_2 = P_G h_2 \in G$$

$$f_1, f_2 \in G^\perp$$

$$h = \alpha_1 h + \alpha_2 h_2$$

$$= \alpha_1 (g_1 + f_1) + \alpha_2 (g_2 + f_2)$$

$$= (\alpha_1 g_1 + \alpha_2 g_2) \leftarrow \in G$$

$$+ (\alpha_1 f_1 + \alpha_2 f_2) \leftarrow \in G^{\perp}$$

$$h = g + f \quad \text{(unique)}$$

$$g = P_G h = \alpha_1 g_1 + \alpha_2 g_2$$

$$= \alpha_1 P_G h_1 + \alpha_2 P_G h_2$$

$$\therefore$$

Idempotency: $(P \equiv P_G)$

$$P^2 = P$$

$h \in \mathcal{H}$

$h = g + f,\ g = P_G h \in G$

$P_G g = g \in G$

$$P_G^2 h = P_G(P_G h)$$

$$= P_G g = g = P_G h$$

$\therefore$

Hermiticity:

$$P = P^{\dagger}$$

$$h_1 = g_1 + f_1 \,, \quad g_1, g_2 \in G$$

$$h_2 = g_2 + f_2 \quad f_1, f_2 \in G^{\perp}$$

$$g_1 = P_G h_1 \equiv P h_1$$

$$g_2 = P_G h_2 \equiv P h_2$$

$$(P h_1, h_2) = (g_1, g_2 + f_2)$$

$$= (g_1, g_2) + \cancel{(g_1, f_2)}^{\to 0}$$

$$= (g_1, g_2) = (g_1, g_2) + \underbrace{(f_1, g_2)}_{0}$$

$$= (g_1 + f_1, g_2)$$

$$= (h_1, P h_2)$$

$\therefore$

$$\|Ph\| \leq \|h\|$$

Proof:

$$\|Ph\|^2 = |(Ph, Ph)| = |(h, P^2 h)|$$

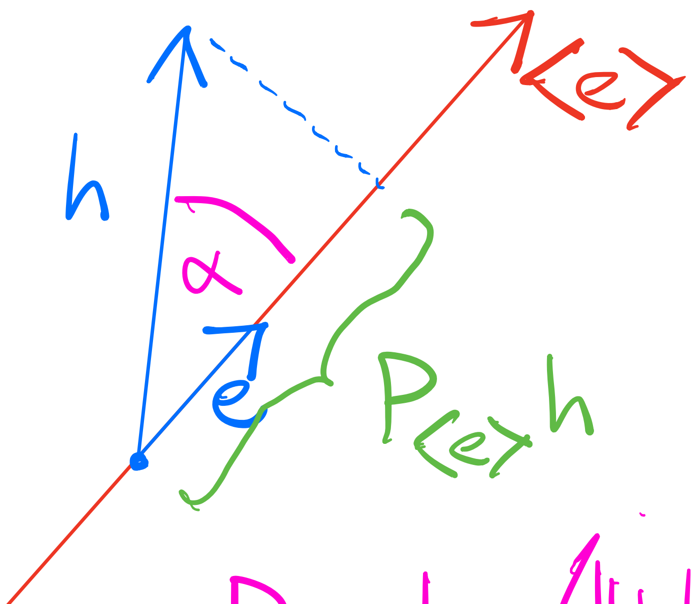
$$= |(h, Ph)| \underset{\text{inequality}}{\overset{\text{Schwartz}}{\leq}} \|h\| \|Ph\|$$

$$\|Ph\| \leq \|h\|$$

$\therefore$

Let us consider $\langle e \rangle$, where $\|e\| = 1$.

$\langle e \rangle \to$ subspace of multiples of e, λe, $\lambda \in \mathbb{C}$.



$$P_{\langle e \rangle} h = \underbrace{\left(\|h\| \cos\alpha \right)}_{(e,h)} e$$

$$P_{\langle e \rangle} h = (e,h)e$$

Proof:

$$P_{\langle e \rangle} h = \lambda e, \quad \lambda \in \mathbb{C}$$

$$(e, P_{\langle e \rangle} h) = (e, \lambda e) = \lambda (e,e) = \lambda \|e\|^2 = \lambda$$

$$\Updownarrow$$

$$(\underbrace{P_{\langle e \rangle}}e, h) = (e,h)$$

$$\lambda = (e,h) \quad \therefore$$

$$P_{\langle e \rangle} h = (e,h)\, e$$

$$P_{\langle e \rangle} |h\rangle = \langle e|h\rangle |e\rangle$$

$$= |e\rangle\langle e|h\rangle$$

$$P_{\langle e \rangle} = |e\rangle\langle e| \quad \checkmark$$

$$P_{\langle e \rangle} |h\rangle = |e\rangle\langle e|h\rangle$$

$$= e\,(e,h) = (e,h)\,e$$

$$P_{\langle e \rangle}^{\dagger} = (|e\rangle\langle e|)^{\dagger}$$

$$= (\langle e|)^{\dagger}(|e\rangle)^{\dagger}$$

$$= |e\rangle\langle e| = P_{\langle e \rangle}$$

$$P_{\langle e \rangle}^{2} = P_{\langle e \rangle} P_{\langle e \rangle}$$

$$= |e\rangle\underbrace{\langle e|e\rangle}_{\|e\|^2 = 1}\langle e|$$

$$= |e\rangle\langle e| = P_{\langle e \rangle}$$