A - operator, $\overline{D(A)} = \mathcal{H}$
(dense in $\mathcal{H}$)

A^{+} - adjoint operator

$$(f, Ag) = (A^{+}f, g)$$

$g \in D(A)$

$f \in D(A^{+})$

A is self-adjoint if $A = A^{+}$

A is hermitian (symmetric) if $(f, Ag) = (Af, g)$ for $f, g \in D(A)$

For hermitian operators,

$$D(A) \subseteq D(A^\dagger)$$

To prove that A is self-adjoint, we must prove that

$(D(A) \subseteq D(A^\dagger))$ ✓

✓ A is hermitian

✓ $D(A^\dagger) = D(A)$ ✓

$(D(A^\dagger) \subseteq D(A))$ ✓

Operators in QM must be self-adjoint

For each self-adjoint operator, its spectrum consists of

eigenvalues

- point spectrum
- continuous spectrum

and is real ($\mathbb{R}$).

In practice, we usually use finite-dimensional Hilbert spaces defined by linear combinations of basis functions (vectors). In this case, domains of operators are the entire Hilbert space and all operators are bounded.

This means that hermitian operators are also self-adjoint.

A, $\{e_1, \dots, e_n\}$ ← orthonormal basis

$$(e_i, e_j) = \delta_{ij} \quad (\|e_i\| = 1)$$

$f \in \mathcal{H}$ $\quad (D(A) = \mathcal{H})$

$$f = \sum_{k=1}^{n} f_k e_k$$

$$Af = \sum_{k=1}^{n} f_k \, A e_k \qquad \checkmark$$

$$A e_k = \sum_{l=1}^{n} A_{lk} \, e_l \qquad \checkmark$$

matrix elements

$$\begin{array}{c} \underline{\underline{A}} \\ \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \cdots & \uparrow \\ Ae_1 & Ae_2 & & Ae_n \end{matrix} \end{array}$$

matrix $\underline{\underline{A}}$ representing operator A

$$Af = \sum_{k=1}^{n} f_k \sum_{l=1}^{n} A_{lk} \, e_l$$

✓ $$Af = \sum_{l=1}^{n} \underbrace{\left(\sum_{k=1}^{n} A_{lk} f_k \right)}_{(\underline{\underline{A}} \cdot \underline{f})_l} e_l$$

$$\boxed{\underline{\underline{A}}} \cdot \boxed{\underline{f}} = \boxed{\underline{Af}}$$

$$\|Af\| \leq M \|f\| \; ?$$

$$\|Af\| \leq \sum_{l=1}^{n} \left| \sum_{k=1}^{n} A_{lk} f_k \right| \underbrace{\|e_k\|}_{1}$$

$$\|Af\| \leq \sum_l \left| \sum_k A_{lk} f_k \right|$$

$$= \sum_l \left| \left(A_{l1} f_1 + \cdots + A_{ln} f_n \right) \right|$$

$$= \sum_l \left| [A_{l1}^{*}, \ldots, A_{ln}^{*}] \bullet [f_1, \ldots, f_n] \right|$$

$$\vec{a} \cdot \vec{b} = \sum_i a_i^{*} b_i$$

Schwartz inequality

$$\leq \sum_l \left(\sqrt{|A_{l1}|^2 + \cdots + |A_{ln}|^2} \times \sqrt{|f_1|^2 + \cdots + |f_n|^2} \right)$$

$$\|f\| = \sqrt{(f, f)}$$

$$= \sqrt{\left(\sum_k f_k e_k, \sum_l f_l e_l\right)}$$

$$= \sqrt{\sum_{k,l} f_k^* f_l \underbrace{(e_k, e_l)}_{\delta_{kl}}}$$

$$= \sqrt{\sum_{k} f_k^* f_k}$$

$$= \sqrt{\sum_{k=1}^{n} |f_k|^2}$$

$$\|Af\| \leq \left(\sum_{l=1}^{n} \sqrt{\sum_{k=1}^{n} |A_{lk}|^2}\right) \|f\|$$

$$\|Af\| \leq M \|f\|$$

$$M = \sum_{\ell=1}^{n} \sqrt{\sum_{k=1}^{n} |A_{\ell k}|^2}$$

$$< \infty$$

A is bounded, $D(A) = \mathcal{H}$

If A is hermitian,

$D(A^\dagger) = \mathcal{H} = D(A)$,

so $A = A^\dagger$, i.e., also self-adjoint.

(i) eigenvalues (point spectra) of hermitian operators are real numbers.

Proof:

eigenvalue

$$Af = \lambda f, \quad f \neq \Theta$$

$$(f, Af) = (f, \lambda f) = \lambda (f, f) = \lambda \|f\|^2$$

TOR

$$(Af, f) = (\lambda f, f) = \lambda^* (f, f) = \lambda^* \|f\|^2$$

$$\lambda \|f\|^2 = \lambda^* \|f\|^2$$

$$f \neq \Theta, \ \|f\| > 0$$

$$\lambda = \lambda^* \in \mathbb{R}$$

(ii) Eigenvectors f_1 and f_2 of a hermitian operator A correspon-ding to different eigenvalues λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$ are orthogonal.

Proof:

$$Af_1 = \lambda_1 f_1 \ , \quad Af_2 = \lambda_2 f_2$$

$$(f_1, Af_2) = (f_1, \lambda_2 f_2) = \underline{\lambda_2 (f_1, f_2)}$$

$\|$ TOR

$$(Af_1, f_2) = (\lambda_1 f_1, f_2) = \underline{\lambda_1^* (f_1, f_2)}$$

$$\lambda_1^* (f_1, f_2) = \lambda_2 (f_1, f_2)$$

(i) $\|$

λ_1

$$\lambda_1(f_1,f_2)=\lambda_2(f_1,f_2)$$

$$(\lambda_1-\lambda_2)(f_1,f_2)=0$$

$$\lambda_1\neq\lambda_2$$

$$(f_1,f_2)=0$$

f_1 is orthogonal to f_2

$$(f_1\perp f_2)$$

(iii) An expression

(f, Af) is

real for every $f\in D(A)$

if A is hermitian.

$$(f, Af) \overset{\text{TOR}}{=} (Af, f)$$

$$\overset{(a)}{=} (f, Af)^*$$

$$(f, Af) \in \mathbb{R}$$

$$\langle \hat{F} \rangle = \int \psi^* \hat{F} \psi \, d\tau$$

$$= (\psi, \hat{F}\psi) \in \mathbb{R}$$

Expectation values of hermitian (and self adjoint) operators are real numbers.