HILBERT SPACE is an infinite or finite-dimensional LINEAR INNER PRODUCT (UNITARY) SPACE WHICH IS ALSO A COMPLETE METRIC SPACE with respect to the metric generated by the INNER (SCALAR) PRODUCT.

(SPACES)

$$
\begin{tikzcd}
\text{LINEAR} \\ (\alpha f+\beta g \in \mathcal{H}) \arrow[dr] & & \text{METRIC} \\ D[f,g] \arrow[dl] \\
& \text{NORM} \\ \|f\| \arrow[r, "\text{complete}"] \arrow[d] & \text{BANACH} \arrow[d] \\
& \text{UNITARY} \\ (f,g) \arrow[r, "\text{complete}"'] & \text{HILBERT}
\end{tikzcd}
$$

METRIC SPACES

$\mathcal{H}$, $f, g, \ldots$

$D[f,g] \geq 0$, distance

(a) $D[f,g] = D[g,f] > 0$ for $f \neq g$

(b) $D[f,f] = 0$

(c) $D[f,g] \leq D[f,h] + D[h,g]$ (triangle inequality)

$\{f_n\} \in \mathcal{H}$ (sequence in $\mathcal{H}$)

$$f_n \xrightarrow[n\to\infty]{\text{converges to}} f \in \mathcal{H}$$

if $D[f_n, f] \xrightarrow[n\to\infty]{} 0$

$$\lim_{n\to\infty} f_n = f$$

$$D[f_n, f_m] \xrightarrow[n,m\to\infty]{} 0$$

$$D[f_n, f_m] \underset{(c)}{\leq} D[f_n, f] + D[f, f_m]$$

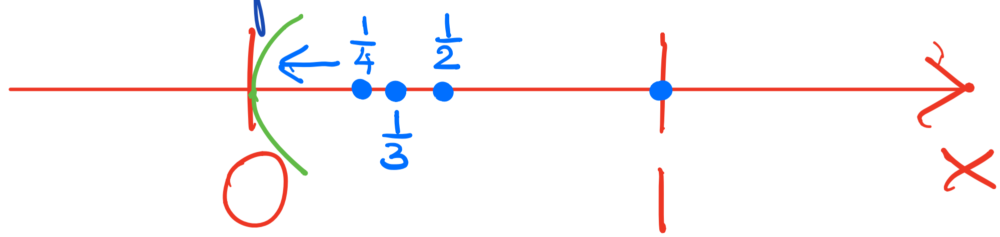
$$\underset{(a)}{=} \underbrace{D[f_n, f]}_{\substack{\downarrow n\to\infty \\ 0}} + \underbrace{D[f_m, f]}_{\substack{\downarrow m\to\infty \\ 0}}$$

$$\xrightarrow[n,m\to\infty]{} 0$$

If f_n converges to f,
$\lim_{n,m\to\infty} D[f_n, f_m] = 0$ } ALWAYS TRUE

But the opposite is not always true

Example:



$$\mathcal{X} = (0,1]$$

$$f_n = \frac{1}{n}$$

$$D[f_n, f_m] = \left|\frac{1}{n} - \frac{1}{m}\right| \xrightarrow[n,m \to \infty]{} 0$$

$$\lim_{n \to \infty} f_n = 0 \notin \mathcal{X}$$

If $D[f_n, f_m] \xrightarrow[n,m\to\infty]{} 0$,
$\{f_n\}$ is called
FUNDAMENTAL

CONVERGENCE $\xrightarrow[\text{true}]{\text{always}}$ FUNDAMENTAL

FUNDAMENTAL $\xrightarrow[\text{always true}]{\text{NOT}}$ CONVERGENCE

$\mathcal{H}$ IS COMPLETE
IF EVERY FUNDAMENTAL
SEQUENCE IN $\mathcal{H}$ CONVERGES
TO AN ELEMENT IN
$\mathcal{H}$.

$G \subset \mathcal{H}$ (subset of $\mathcal{H}$)

$\overline{G} \equiv G$ plus all limits of sequencies in G

CLOSURE OF G

(e.g., $G = (0,1]$; $\overline{G} = [0,1]$)

If $G = \overline{G}$, then

G IS CLOSED

($[0,1]$ is closed)

If $\overline{G} = \mathcal{H}$, then

G IS DENSE IN $\mathcal{H}$.

NORM SPACES

Norm space is a linear space in which for each vector f we can define a real nonnegative number $\|f\|$, called a norm of f, which satisfies

triangle inequality

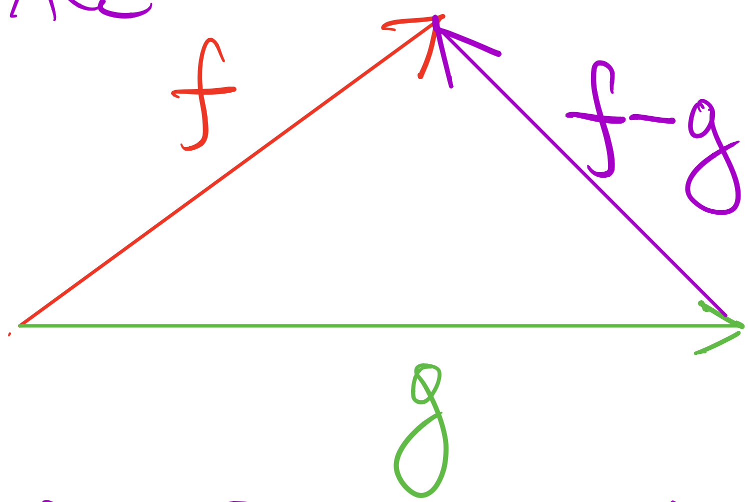
(a) $\|\alpha f\| = |\alpha| \, \|f\|$

(b) $\|f+g\| \leqslant \|f\| + \|g\|$

(c) $\|f\| = 0 \Rightarrow f = \Theta$

zero vector

EVERY NORM SPACE IS ALSO A METRIC SPACE



Define: $D[f,g] = \|f-g\|$ ✓

(a) $D[f,g] = \|f-g\|$

$= \|(-1)(g-f)\|$

$\overset{\text{(a) of norm}}{=} |-1|\,\|g-f\|$

$= \|g-f\| = D[g,f]$

(b) $D[f,f] = \|f-f\|$

$= \|\Theta_{ff}\| = 0$

$(\|\Theta_{ff}\| = \|0 \cdot g\| = |0|\,\|g\|$

$= 0)$

(c) $D[f,g] = \|f-g\|$

$= \|(f-h)+(h-g)\|$

$\overset{(b)}{\leq} \|f-h\| + \|h-g\|$

of norm

$= D[f,h] + D[h,g]$

Example: $\mathbb{R}^n$, $x \in \mathbb{R}^n$

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

A complete norm space (complete with respect to $D[f,g] = \|f-g\|$) is called a

BANACH SPACE

UNITARY SPACES

A linear space $\mathcal{H}$ is unitary if for every pair $f, g \in \mathcal{H}$, there exists a real or complex number (f, g), which satisfies

inner or scalar product of f and g

(a) $(g,f) = (f,g)^*$

(b) $(f, \alpha_1 g_1 + \alpha_2 g_2)$

(bilinear) $= \alpha_1 (f,g_1) + \alpha_2 (f,g_2)$

$(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1^* (f_1,g)$

$+ \alpha_2^* (f_2,g)$

Proof of $\nearrow$

$$(\alpha_1 f_1 + \alpha_2 f_2, g) \overset{(a)}{=} (g, \alpha_1 f_1 + \alpha_2 f_2)^*$$

$$\overset{\text{purple (b)}}{=\!=} \left[\alpha_1 (g,f_1) + \alpha_2 (g,f_2)\right]^*$$

$$= \alpha_1^* (g,f_1)^* + \alpha_2^* (g,f_2)^*$$

$$\overset{\text{purple (a)}}{=\!=} \alpha_1^* (f_1,g) + \alpha_2^* (f_2,g)$$

(c) $(f,f) \geq 0$ (real)

$(f,f) = 0 \Rightarrow f = \Theta$

$$(f,g) \xrightarrow[\text{scalar product}]{\text{inner product}} \mathbb{C}$$ ← complex numbers

$$\mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$$

Example: $\mathbb{C}^n$

$(x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{C}$, $i = 1, \ldots, n$

$$(x,y) = x \cdot y \quad \checkmark$$

$$= \sum_{i=1}^{n} x_i^* y_i$$ ← satisfies (a), (b), (c)

$\boxed{\text{Every unitary space is also a norm space.}}$

Define $\|f\| = \sqrt{(f,f)}$ ✓

Example: $\mathbb{C}^n$

$$(x,y) = \sum_{i=1}^{n} x_i^* y_i$$

$$\|x\| = \sqrt{(x,x)} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

We will now prove that $\|f\| = \sqrt{(f,f)}$ is a properly defined norm.

Proof:

(a) $\|\alpha f\| = \sqrt{(\alpha f, \alpha f)}$

$\underset{\text{of scalar product}}{\overset{(b)}{=}} \sqrt{\alpha^* \alpha (f,f)}$

$$= \sqrt{|\alpha|^2 (f,f)}$$

$$= |\alpha| \sqrt{(f,f)}$$

$$= |\alpha| \, \|f\|$$

(c) $\|f\| = 0 \Longrightarrow$

$$\sqrt{(f,f)} = 0$$

$$\Longrightarrow (f,f) = 0$$

$$\xRightarrow[\text{of scalar product}]{(c)} f = \Theta$$

(b) We must prove that

$$\|f+g\| \leq \|f\| + \|g\| \quad \checkmark$$

for $\|f\| = \sqrt{(f,f)}$.

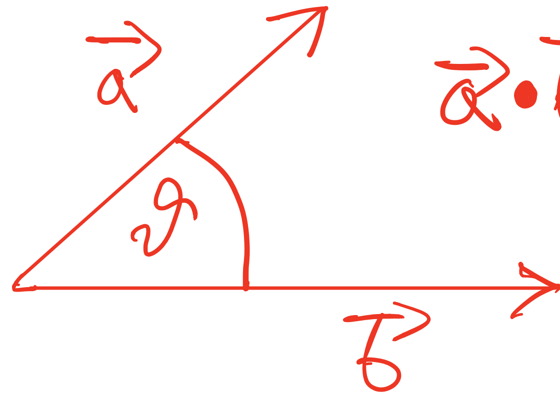
Schwartz inequality.

$$|(f,g)| \leq \|f\| \, \|g\| \quad \checkmark$$

We can have equality

$$|(f,g)| = \|f\| \, \|g\|$$

iff f and g are linearly dependent.



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \times \cos\vartheta$$

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \, \|\vec{b}\|$$

Proof: $g = \Theta$ or $g \neq \Theta$

$g = \Theta \Longrightarrow$ obvious $\left((f, \Theta) = 0, \ \|\Theta\| = 0 \right)$

Consider $g \neq \Theta$.

$$0 \overset{(c)}{\leq} (f + \alpha g, f + \alpha g)$$

$$\overset{(b)}{=} (f, f) + \alpha (f, g) + \alpha^* (g, f) + \alpha^* \alpha (g, g)$$

$$= \|f\|^2 + \alpha (f,g) + \alpha^* (g,f) + |\alpha|^2 \|g\|^2$$

true for ANY $\alpha \in \mathbb{C}$

Let us use

$$\alpha = - \frac{(g,f)}{\|g\|^2}$$

$$\alpha^* = - \frac{(f,g)}{\|g\|^2} \quad (g,f)^*$$

$$(f,g)^* \;\|$$

$$0 \leq \|f\|^2 - \frac{(g,f)(f,g)}{\|g\|^2}$$

$$- \frac{(f,g)(g,f)}{\|g\|^2} \quad = (f,g)^*$$

$$+\frac{|(f,g)|^2}{\|g\|^{\cancel{4}2}}\cancel{\|g\|^2}$$

$$=\|f\|^2-\frac{|(f,g)|^2}{\|g\|^2}$$

$$\cancel{-\frac{|(f,g)|^2}{\|g\|^2}}\;\cancel{+\frac{|(f,g)|^2}{\|g\|^2}}$$

$$\|f\|^2-\frac{|(f,g)|^2}{\|g\|^2}\geq 0$$

$$|(f,g)|^2\leq\|f\|^2\|g\|^2$$

$$|(f,g)|\leq\|f\|\,\|g\| \quad \therefore$$

We get equality only if

$$0 = (f + \alpha g, f + \alpha g)$$

$$\Rightarrow f + \alpha g = 0$$

$$\Rightarrow f = -\alpha g$$

(linearly dependent)

$\therefore$

We are now ready to return to the triangle inequality for the norm.

$$\|f+g\|^2 = (f+g, f+g)$$

$$= (f,f) + (f,g) + \underbrace{(g,f)}_{=(f,g)^*}$$

$$+ (g,g) = \|f\|^2$$

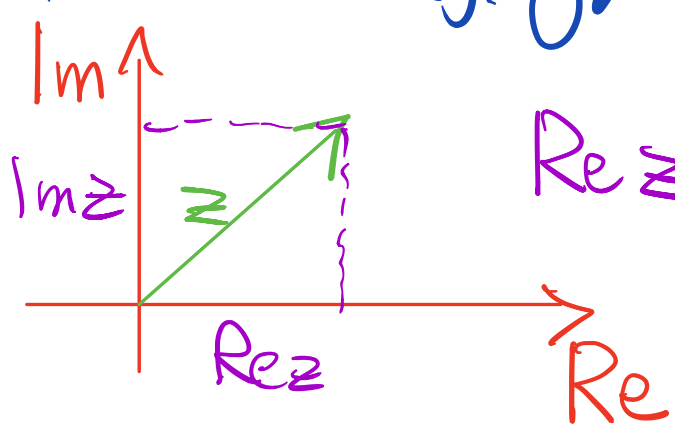
$$+ \|g\|^2$$

our z here is (f,g)

$$\left(z + z^* = 2\,\mathrm{Re}\, z,\ z \in \mathbb{C} \right)$$

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2$$

$$+ 2\,\mathrm{Re}\,(f,g)$$



$$\mathrm{Re}\, z \leq |z|$$

$$\|f+g\|^2 \leq \|f\|^2 + \|g\|^2 + 2\,|(f,g)|$$

Schwartz inequality

$$\leq \|f\|^2 + \|g\|^2 + 2\,\|f\|\,\|g\|$$

$$= (\|f\| + \|g\|)^2$$

$$\|f+g\|^2 \leq (\|f\| + \|g\|)^2$$

$$\|f+g\| \leq \|f\| + \|g\|$$

(b) proved $\therefore$

Unitary space is a space with a norm

$$\|f\| = \sqrt{(f,f)}$$ ✓

and, also, a metric space

$$D[f,g] = \|f-g\|$$ ✓

$$= \sqrt{(f-g,f-g)}$$

Example: $\mathbb{C}^n$

$x = (x_1, \ldots x_n) \in \mathbb{C}^n$, $x_i \in \mathbb{C}$, $i=1,\ldots,n$

$$(x,y) = x \cdot y = \sum_{i=1}^{n} x_i^* y_i$$

$$\|x\| = \sqrt{(x,x)} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$D[x,y] = \|x-y\|$$

$$= \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

HILBERT SPACE IS

- a linear space, $\alpha f + \beta g \in \mathscr{H}$ if $f, g \in \mathscr{H}$
- unitary, $(f,g) \in \mathbb{C}$
- metric, $\|f\| = \sqrt{(f,f)}$, $D[f,g] = \|f-g\|$

- complete with respect to $D[f,g]$

↓complete unitary space

HILBERT SPACE IS ALSO A BANACH SPACE

↑complete norm space

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f and g are orthogonal if $(f,g)=0$.

A set $\{f_1, \ldots, f_n\}$ is orthogonal if $\forall$ $(f_i, f_j) = 0$ $i \neq j$

A set $\{f_1, \dots, f_n\}$ is orthonormal if

$$(f_i, f_j) = \delta_{ij} \quad \forall i, j$$

[$\{f_1, \dots, f_n\}$ is orthogonal and $(f_i, f_i) = 1$
$\Longrightarrow \|f_i\| = 1, \; i = 1, \dots, n$]

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① $(f, g) = 0 \quad \forall g \in \mathcal{H}$ implies that $f = \Theta$.

Proof:
$(f, g) = 0$, use $g = f$

$(f,f)=0 \Rightarrow f=\Theta$.

② If $(f,g)=0$ ✓
for every $g \in G$
and G is dense in $\mathcal{H}$
$(\overline{G}=\mathcal{H})$, then
$f=\Theta$.

Proof:

arbitrary g in $\mathcal{H}$

Consider $g \in \mathcal{H}$. ✓
Since G is dense
in $\mathcal{H}$, g must be a
limit of some sequence
$\{g_n\}$ from G, $g_n \in G$.

$$\begin{array}{ccc} g_n & \xrightarrow[n\to\infty]{} & g \\ \cap & & \\ G & & \end{array}$$

✓ $(f, g_n) = 0$, since $g_n \in G$.

✓ $(f, g_n) \xrightarrow[n\to\infty]{} (f, g)$, since

$$|(f, g_n) - (f, g)|$$

$$= |(f, g_n - g)|$$

$$\leq \|f\| \, \|g_n - g\| \xrightarrow[n\to\infty]{} 0$$

$$0 = \lim_{n\to\infty} (f, g_n) = (f, g)$$

$$(f,g)=0 \ \forall_{g\in\mathcal{H}} \overset{①}{\Longrightarrow} f=\Theta \ \therefore$$

③ If $(f_1,g)=(f_2,g)$

for each $g\in G$,

where G is dense

in $\mathcal{H}$ $(\overline{G}=\mathcal{H})$,

$f_1=f_2$.

Proof: $(f_1,g)=(f_2,g)$

$$(f_1,g)-(f_2,g)=0$$

$$(f_1-f_2,g)=0$$

Use ②, $f_1 - f_2 = ⊕$,

$f_1 = f_2$ $\therefore$