CEM 991 **Fall 2021**

Assignment No. 7 (25 pts.) **Due Date: December 1, 2021 (Wednesday)**

1. (13 pts.) Let F and G be two dynamical variables and let $\hat{F}$ and $\hat{G}$ represent the corresponding quantum-mechanical operators. Let $|\phi\rangle$ be the normalized quantum state and let $\langle\hat{F}\rangle$ and $\langle\hat{G}\rangle$ represent the expectation values of F and G in the state described by $|\phi\rangle$. Let us define the following operators:

$$\Delta\hat{F} = \hat{F} - \langle\hat{F}\rangle, \tag{1}$$

$$\Delta\hat{G} = \hat{G} - \langle\hat{G}\rangle, \tag{2}$$

$$\hat{A} = \Delta\hat{F} + \imath\,\alpha\,\Delta\hat{G}, \tag{3}$$

where α is a real variable and $\imath$ is an imaginary unit. Let us also introduce the auxiliary function

$$I(\alpha) = \langle\hat{A}^{\dagger}\hat{A}\rangle = ||\hat{A}|\phi\rangle||^2. \tag{4}$$

As discussed in class, the condition

$$I(\alpha) \geq 0, \quad \text{for all } \alpha, \tag{5}$$

leads to the Heisenberg relation,

$$(\Delta F)^2\,(\Delta G)^2 \geq -\frac{1}{4}\langle[\hat{F},\hat{G}]\rangle^2, \tag{6}$$

where ΔF and ΔG are the uncertainties of F and G in a state described by $|\phi\rangle$.

(a) [5 pts.] Let us assume that $|\phi\rangle$ is a *minimum wave packet*, so that

$$(\Delta F)^2\,(\Delta G)^2 = -\frac{1}{4}\langle[\hat{F},\hat{G}]\rangle^2 \tag{7}$$

in a quantum state described by $|\phi\rangle$. Show that Eq. (7) is equivalent to the condition

$$I(\alpha_0) = 0, \tag{8}$$

where

$$\alpha_0 = -\frac{\imath\langle[\hat{F},\hat{G}]\rangle}{2(\Delta G)^2}. \tag{9}$$

(b) [3 pts.] Show that condition (8), with α_0 defined by Eq. (9), is equivalent to an equation

$$\left[\Delta\hat{F} + \frac{\langle[\hat{F},\hat{G}]\rangle}{2(\Delta G)^2}\Delta\hat{G}\right]|\phi\rangle = 0. \tag{10}$$

Equation (10) can be viewed as a fundamental equation determining the *minimum wave packet* $|\phi\rangle$.

(c) [5 pts.] Let us consider the one-dimensional *minimum wave packet* $|\phi\rangle$ for $F = p_x$ and $G = x$, so that

$$\Delta x \, \Delta p_x = \frac{\hbar}{2}. \tag{11}$$

Show that Eq. (10) for $F = p_x$ and $G = x$ is equivalent to a differential equation

$$\left[\frac{d}{dx} + \frac{x - x_0}{2(\Delta x)^2} - \frac{\imath}{\hbar} p_0 \right] \phi(x) = 0, \tag{12}$$

where $x_0 = \langle \hat{x} \rangle$ and $p_0 = \langle \hat{p}_x \rangle$. Find the solution of Eq. (12) (you are not required to normalize $\phi(x)$).

2. (8 pts.) One can show that for any eigenstate u_n of the linear harmonic oscillator,

$$\langle \hat{V} \rangle_n \equiv \langle u_n | \hat{V} | u_n \rangle = \tfrac{1}{2} E_n = \tfrac{1}{2} \hbar \omega_c (n + \tfrac{1}{2}). \tag{13}$$

Here, $\hat{V} = \frac{1}{2} K \hat{x}^2$ is the potential energy operator, $K = m\omega_c^2$ is the force constant, ω_c is the angular frequency of the classical harmonic oscillator, and E_n is the energy of the n-th state u_n. Use Eq. (13) and the fact that $\langle \hat{x} \rangle_n \equiv \langle u_n | \hat{x} | u_n \rangle$ and $\langle \hat{p_x} \rangle_n \equiv \langle u_n | \hat{p_x} | u_n \rangle$ vanish to show that the uncertainty product $\Delta x \, \Delta p_x$ equals $(n + \frac{1}{2})\hbar$ for the state described by u_n. What can you say about the $n = 0$ state from the point of view of Problem 1?

3. (4 pts.) Show that the following operator identity is true,

$$Q - \frac{d}{dQ} = -e^{\frac{1}{2}Q^2} \frac{d}{dQ} \, e^{-\frac{1}{2}Q^2}, \tag{14}$$

where Q is a dimensionless variable.

SOLUTIONS

1. (a) As discussed in class,

$$I(\alpha) = \langle \hat{A}^\dagger \hat{A} \rangle = \langle (\Delta\hat{F} - i\alpha\Delta\hat{G})(\Delta\hat{F} + i\alpha\Delta\hat{G}) \rangle$$

$$= \langle (\Delta\hat{F})^2 + i\alpha(\Delta\hat{F})(\Delta\hat{G}) - i\alpha(\Delta\hat{G})(\Delta\hat{F})$$

$$+ \alpha^2(\Delta\hat{G})^2 \rangle = \langle (\Delta\hat{F})^2 \rangle$$

$$+ i\alpha \langle [\Delta\hat{F}, \Delta\hat{G}] \rangle + \alpha^2 \langle (\Delta\hat{G})^2 \rangle,$$

where $\langle\ \rangle$ is the expectation value in the state described by $|\varphi\rangle$.

Since $[\Delta\hat{F}, \Delta\hat{G}] = [\hat{F}, \hat{G}]$ and

$(\Delta F)^2 = \langle (\Delta\hat{F})^2 \rangle$, $(\Delta G)^2 = \langle (\Delta\hat{G})^2 \rangle$,

we obtain

$$I(\alpha) = a\alpha^2 + b\alpha + c, \quad \text{where}$$

$$a = (\Delta G)^2, \quad b = i\langle [\hat{F}, \hat{G}] \rangle, \quad c = (\Delta F)^2.$$

We know that $I(\alpha) \geqslant 0$ for all α, so that the discriminant $\Delta = b^2 - 4ac \leqslant 0$.

$$\Delta = -\langle [\hat{F},\hat{G}] \rangle^2 - 4(\Delta F)^2(\Delta G)^2 \leqslant 0$$

implies that

$$(\Delta F)^2(\Delta G)^2 \geqslant -\frac{1}{4}\langle [\hat{F},\hat{G}] \rangle^2,$$

which is the Heisenberg uncertainty principle.

If we want to obtain the equality,

$$(\Delta F)^2(\Delta G)^2 = -\frac{1}{4}\langle [\hat{F},\hat{G}] \rangle^2,$$

we must assume that $\Delta = 0$. In that case, the quadratic form $I(\alpha) \geqslant 0$ and $I(\alpha) = 0$ only at one point,

$$\alpha = \alpha_0 = -\frac{b}{2a} = -\frac{i\langle [\hat{F},\hat{G}] \rangle}{2(\Delta G)^2}.$$

At $\alpha = \alpha_0$, $I(\alpha_0) = 0$.

Let us calculate $I(\alpha_0)$ from the definition of $I(\alpha)$:

$$I(\alpha_0) = a\alpha_0^2 + b\alpha_0 + c$$

$$= (\Delta G)^2 \alpha_0^2 + i\langle[\hat{F},\hat{G}]\rangle \alpha_0 + (\Delta F)^2$$

$$= -(\Delta G)^2 \frac{\langle[\hat{F},\hat{G}]\rangle^2}{4(\Delta G)^4} + \frac{\langle[\hat{F},\hat{G}]\rangle^2}{2(\Delta G)^2}$$

$$+ (\Delta F)^2 = \frac{1}{4}\frac{\langle[\hat{F},\hat{G}]\rangle^2}{(\Delta G)^2} + (\Delta F)^2.$$

We know that $I(\alpha_0) = 0$. Thus,

$$\frac{1}{4}\frac{\langle[\hat{F},\hat{G}]\rangle^2}{(\Delta G)^2} + (\Delta F)^2 = 0, \text{ or}$$

$$(\Delta F)^2(\Delta G)^2 = -\frac{1}{4}\langle[\hat{F},\hat{G}]\rangle^2,$$

which completes the solution of problem 1 (a).

(b) $I(\alpha_0) = 0$ implies that

$$\langle \hat{A}^{+}_{\alpha_0}\hat{A}_{\alpha_0}\rangle = \| \hat{A}_{\alpha_0}|\varphi\rangle\|^2 = 0,$$

where $$\hat{A}_{\alpha_0} = \Delta\hat{F} + i\alpha_0 \Delta\hat{G}.$$

Thus, $$\hat{A}_{\alpha_0}|\varphi\rangle = (\Delta\hat{F} + i\alpha_0\Delta\hat{G})|\varphi\rangle$$
$$= 0 \quad \text{(zero vector)}.$$

From the formula for α_0, Eq. (9), we obtain

$$\left[\Delta\hat{F} + i\left(-\frac{i\langle[\hat{F},\hat{G}]\rangle}{2(\Delta G)^2}\right)\Delta\hat{G}\right]|\varphi\rangle = 0,$$

$$\left[\Delta\hat{F} + \frac{\langle[\hat{F},\hat{G}]\rangle}{2(\Delta G)^2}\Delta\hat{G}\right]|\varphi\rangle = 0.$$

(c) Let us assume that $\hat{F} = \hat{p}_x$, $\hat{G} = \hat{x}$, and consider the minimum wave packet $|\varphi\rangle$, such that $\Delta x \, \Delta p_x = \frac{\hbar}{2}$.

From Eq. (10), we can write that $\varphi \equiv \varphi(x)$ satisfies the equation

$$\left(\Delta \hat{p}_x + \frac{1}{2} \frac{(-i\hbar)}{(\Delta x)^2} \Delta \hat{x} \right) \varphi(x) = 0,$$

where we used the fact that $[\hat{p}_x, \hat{x}] = -i\hbar$ and the fact that φ is normalized.

Define $x_0 \equiv \langle \hat{x} \rangle$, $p_0 = \langle \hat{p}_x \rangle$.

$$\Delta \hat{p}_x = \hat{p}_x - p_0 = \frac{\hbar}{i} \frac{d}{dx} - p_0 ,$$

$$\Delta \hat{x} = \hat{x} - x_0 .$$

We obtain,

$$\left[\frac{\hbar}{i} \frac{d}{dx} - p_0 - \frac{i\hbar}{2(\Delta x)^2} (x - x_0) \right] \varphi = 0$$

or (after multiplying by $\frac{i}{\hbar}$),

$$\left[\frac{d}{dx} + \frac{x - x_0}{2(\Delta x)^2} - \frac{i}{\hbar} p_0 \right] \varphi(x) = 0,$$

which is equation (12). We can rewrite Eq. (12) as

$$\frac{d\varphi}{dx} + \left[\frac{x-x_0}{2(\Delta x)^2} - \frac{i}{\hbar} p_0\right]\varphi = 0.$$

This is an equation with separable variables,

$$\frac{d\varphi}{dx} = \left[-\frac{(x-x_0)}{2(\Delta x)^2} + \frac{i}{\hbar} p_0\right]\varphi$$

$$\int \frac{d\varphi}{\varphi} = -\int \frac{(x-x_0)}{2(\Delta x)^2}\, dx + \frac{i}{\hbar} p_0 \int dx$$

$$\ln|\varphi| = -\frac{(x-x_0)^2}{4(\Delta x)^2} + \frac{i}{\hbar} p_0 x + \text{const.}$$

$$\varphi(x) = N \exp\left[-\frac{(x-x_0)^2}{4(\Delta x)^2} + \frac{i p_0 x}{\hbar}\right],$$

where N is the normalization factor. This is the desired solution of Eq. (12).

2. Recall that

$$(\Delta x)^2 = \left\langle \left(\hat{x} - \langle \hat{x} \rangle_n \right)^2 \right\rangle_n ,$$

$$(\Delta p)^2 = \left\langle \left(\hat{p}_x - \langle \hat{p}_x \rangle_n \right)^2 \right\rangle_n .$$

In our case,

$$\langle \hat{x} \rangle_n = 0, \quad \langle \hat{p}_x \rangle_n = 0 ,$$

so that

$$(\Delta x)^2 = \langle (\hat{x})^2 \rangle_n , \quad (\Delta p)^2 = \langle \hat{p}_x^2 \rangle_n .$$

Clearly,

$$\langle \hat{x}^2 \rangle_n = \frac{2}{K} \langle \hat{V} \rangle_n =$$

$$= \frac{\cancel{2}}{m\omega_c^2} \frac{1}{\cancel{2}} \hbar\omega_c \left(n + \frac{1}{2} \right) = \frac{\hbar}{m\omega_c} \left(n + \frac{1}{2} \right),$$

$$\langle \hat{p}_x^2 \rangle = 2m \langle \hat{T} \rangle_n \overset{\langle \hat{T} \rangle_n = \langle \hat{V} \rangle_n = \frac{1}{2} E_n}{=\!=\!=} $$

$$= 2m \langle \hat{V} \rangle_n = \cancel{2}m \cdot \frac{1}{\cancel{2}} \hbar \omega_c \left(n + \frac{1}{2}\right)$$

$$= \hbar \omega_c m \left(n + \frac{1}{2}\right).$$

As a result,

$$(\Delta x)^2 = \langle \hat{x}^2 \rangle_n = \frac{\hbar}{m\omega_c} \left(n + \frac{1}{2}\right),$$

$$(\Delta p_x)^2 = \langle \hat{p}_x^2 \rangle_n = \hbar \omega_c m \left(n + \frac{1}{2}\right),$$

$$(\Delta x)^2 (\Delta p_x)^2 = \frac{\hbar}{\cancel{m\omega_c}} \left(n + \frac{1}{2}\right) \hbar \cancel{\omega_c m} \left(n + \frac{1}{2}\right)$$

$$= \left[\hbar \left(n + \frac{1}{2}\right)\right]^2, \quad \text{so that}$$

$$\Delta x \, \Delta p_x = \left(n + \frac{1}{2}\right) \hbar.$$

For $n=0$, we obtain

$$\Delta x \, \Delta p_x = \frac{\hbar}{2},$$

so that the $n=0$ state $u_0(x)$ is the minimum wave packet. As a matter of fact, the solution $\varphi(x)$ for a minimum wave packet taken from Problem 1(c),

$$\varphi(x) = N \exp\left[-\frac{(x-x_0)^2}{4(\Delta x)^2} + \frac{i p_0 x}{\hbar}\right],$$

becomes equivalent to $u_0(x)$, when adopted to the harmonic oscillator case. Indeed, for $u_0(x)$, $x_0 = 0$, $p_0 = 0$, $(\Delta x)^2 = \frac{\hbar}{2m\omega_c}$, so that

$$\varphi(x) = N \exp\left[-\frac{m\omega_c}{2\hbar} x^2\right] = N e^{-\frac{1}{2}\alpha^2 x^2},$$

where $\alpha = \left(\frac{m\omega_c}{\hbar}\right)^{\frac{1}{2}}$.

3. Consider an arbitrary function $f = f(Q)$.

$$e^{\frac{1}{2}Q^2} \frac{d}{dQ} e^{-\frac{1}{2}Q^2} f(Q) =$$

$$= e^{\frac{1}{2}Q^2} \Big[-\frac{1}{\cancel{2}} \cdot \cancel{2} Q\, e^{-\frac{1}{2}Q^2} f(Q) +$$

$$+ e^{-\frac{1}{2}Q^2} \frac{df}{dQ} \Big] = - Q f(Q) +$$

$$+ \frac{df}{dQ} = -\left(Q - \frac{d}{dQ}\right) f(Q).$$

Thus,

$$Q - \frac{d}{dQ} = - e^{\frac{1}{2}Q^2} \frac{d}{dQ} e^{-\frac{1}{2}Q^2}.$$

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