

1. (4 pts.) Use the following representation of the Dirac delta function [cf. E. Merzbacher, *Quantum Mechanics*, 3rd edition (Wiley, New York, 1998), Appendix, pp. 630-634]:

$$\delta(y) = \lim_{n \rightarrow \infty} \frac{\sin ny}{\pi y},$$

to show that

$$\int_{-\infty}^{+\infty} e^{ikx} dx = 2\pi\delta(k).$$

$$\text{Hint: } \int_{-\infty}^{+\infty} e^{ikx} dx = \lim_{a \rightarrow \infty} \int_{-a}^{+a} e^{ikx} dx.$$

2. (8 pts.) Let $\delta(x)$ be the Dirac delta function. Show that

(a) $\delta(x) = \delta(-x)$.

(b) $x\delta(x) = 0$.

(c) $\delta(ax) = a^{-1}\delta(x)$, $a > 0$.

(d) $\int_{-\infty}^{\infty} \delta(a-x)\delta(x-b) dx = \delta(a-b)$.

Recall that the Dirac delta function is not an ordinary function. The meaning of each of the above equations is that the corresponding left- and right-hand sides, when used as multiplying factors under an integral sign, lead to the same result. For example, in order to prove (a), i.e., $\delta(x) = \delta(-x)$, you should prove that $\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(x)\delta(-x)dx$ for any function $f(x)$.

3. (8 pts.) The z component of the angular momentum operator $\hat{\mathbf{L}}$ is represented by the operator

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi},$$

where ϕ is a polar angle. The normalized eigenfunctions of $\hat{L}_z$ are

$$\phi_m = \frac{1}{\sqrt{2\pi}} e^{im\phi},$$

and the corresponding eigenvalues are $m\hbar$, where $m = 0, \pm 1, \pm 2, \dots$. Consider a state described by the wave function

$$\psi(\phi) = C \cos^2 \phi, \tag{1}$$

where C is a constant. Calculate the possible values $m\hbar$ of L_z and their probabilities $P(m\hbar)$ in a state described by the wave function $\psi(\phi)$, Eq. (1). Calculate the expectation value

$$\langle \hat{L}_z \rangle = \frac{\int_0^{2\pi} \psi^*(\phi) \hat{L}_z \psi(\phi) d\phi}{\int_0^{2\pi} \psi^*(\phi) \psi(\phi) d\phi}$$

and compare the result with the expectation value of $\hat{L}_z$ defined as

$$\langle \hat{L}_z \rangle = \sum_m m\hbar P(m\hbar).$$

Hint: Use the equations $\cos^2 \phi = \frac{1}{2}(1 + \cos 2\phi)$ and $\cos \alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$.

4. (5 pts.) Consider a one-particle system in a state described by a spherically-symmetric wave function $\psi = \psi(r)$ (r is the distance between particle and the origin). Show that ψ is a simultaneous eigenfunction of all three angular momentum operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$, in spite of the fact that $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ do not commute. Find the corresponding eigenvalues. *This example demonstrates that non-commuting operators may have common eigenstates.*

SOLUTIONS

$$1. \quad \int_{-\infty}^{+\infty} e^{ikx} dx = \lim_{a \rightarrow \infty} \int_{-a}^{+a} e^{ikx} dx$$

$$= \lim_{a \rightarrow \infty} \frac{e^{ika} - e^{-ika}}{ik}$$

$$= 2 \lim_{a \rightarrow \infty} \frac{e^{ika} - e^{-ika}}{2ik}$$

From the Euler formula for $e^{\pm ika}$,

$$\frac{e^{ika} - e^{-ika}}{2i} = \sin ka.$$

Thus,

$$\int_{-\infty}^{+\infty} e^{ikx} dx = 2 \lim_{a \rightarrow \infty} \frac{\sin ka}{k}$$

$$= 2\pi \lim_{a \rightarrow \infty} \frac{\sin ka}{\pi k} = \underline{\underline{2\pi \delta(k)}}.$$

2. (a) We must prove that

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = \int_{-\infty}^{\infty} f(x) \delta(-x) dx.$$

The left-hand side (LHS) is

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0).$$

The right-hand side (RHS) is

$$\int_{-\infty}^{\infty} f(x) \delta(-x) dx \quad \begin{array}{l} y = -x \\ dy = -dx \end{array}$$

$$= - \int_{+\infty}^{-\infty} f(-y) \delta(y) dy$$

$$= \int_{-\infty}^{\infty} f(-y) \delta(y) dy = f(0).$$

$$\text{LHS} = \text{RHS}.$$

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(b) We must prove that

$$\int_{-\infty}^{\infty} f(x) x \delta(x) dx = 0.$$

Let us define

$$g(x) = x f(x).$$

We have, $\int_{-\infty}^{\infty} g(x) \delta(x) dx = g(0),$

where $g(0) = 0 \cdot f(0) = 0.$

Thus, $\int_{-\infty}^{\infty} g(x) \delta(x) dx = \int_{-\infty}^{\infty} f(x) x \delta(x) dx$
 $= 0$ for all functions $f(x).$

(c) We must prove that

$$\int_{-\infty}^{\infty} f(x) \delta(ax) dx = \frac{1}{a} \int_{-\infty}^{\infty} f(x) \delta(x) dx$$

$$\text{LHS: } \int_{-\infty}^{\infty} f(x) \delta(ax) dx \quad \begin{array}{l} \underline{y=ax} \\ dy=adx, a>0 \end{array}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f\left(\frac{y}{a}\right) \delta(y) dy$$

$$= \frac{1}{a} f\left(\frac{0}{a}\right) = \frac{1}{a} f(0).$$

RHS:

$$\frac{1}{a} \int_{-\infty}^{\infty} f(x) \delta(x) dx = \frac{1}{a} f(0).$$

$$\underline{\text{LHS} = \text{RHS}}$$

(d) We must prove that

$$\int_{-\infty}^{\infty} f(a) \left(\int_{-\infty}^{\infty} \delta(a-x) \delta(x-b) dx \right) da \\ = \int_{-\infty}^{\infty} f(a) \delta(a-b) da$$

$$\text{LHS: } \int_{-\infty}^{\infty} f(a) \left(\int_{-\infty}^{\infty} \delta(a-x) \delta(x-b) dx \right) da$$

$$= \int_{-\infty}^{\infty} \delta(x-b) \left(\int_{-\infty}^{\infty} f(a) \delta(a-x) da \right) dx$$

$$= \int_{-\infty}^{\infty} \delta(x-b) f(x) dx = f(b).$$

$$\text{RHS: } \int_{-\infty}^{\infty} f(a) \delta(a-b) da = f(b).$$

$$\boxed{\text{LHS} = \text{RHS}}$$

3. In general, if we have an operator $\hat{A}$ and know its eigenvalues a_m and eigenfunctions u_m , we can always write

$$\psi = \sum_m c_m u_m,$$

for any state ψ (let us assume that ψ is normalized). The possible values of $\hat{A}$ in a state described by ψ are characterized by the probabilities $P(a_m) = |c_m|^2$.

Clearly,
$$\sum_m P(a_m) = 1.$$

In this problem, we have:

$$\hat{A} = \hat{L}_z,$$

$$u_m = \frac{1}{\sqrt{2\pi}} e^{im\phi},$$

$$a_m = m\hbar \quad (m = 0, \pm 1, \pm 2, \dots),$$

and

$$\psi(\varphi) = C \cos^2 \varphi.$$

Using the formula

$$\cos^2 \varphi = \frac{1 + \cos 2\varphi}{2},$$

we obtain

$$\begin{aligned} \psi(\varphi) &= C \left(\frac{1}{2} + \frac{1}{2} \cos 2\varphi \right) = \\ &= C \left(\frac{1}{2} + \frac{1}{4} e^{2i\varphi} + \frac{1}{4} e^{-2i\varphi} \right) \\ &= \sqrt{\frac{\hbar}{2}} C \underbrace{\frac{1}{\sqrt{2\hbar}}}_{m=0} + \frac{1}{2} \sqrt{\frac{\hbar}{2}} C \underbrace{\frac{1}{\sqrt{2\hbar}} e^{2i\varphi}}_{m=2} \\ &\quad + \frac{1}{2} \sqrt{\frac{\hbar}{2}} C \underbrace{\frac{1}{\sqrt{2\hbar}} e^{-2i\varphi}}_{m=-2} \end{aligned}$$

$$= c_0 u_0 + c_2 u_2 + c_{-2} u_{-2},$$

where

$$c_0 = \sqrt{\frac{\hbar}{2}} C, \quad c_2 = c_{-2} = \frac{1}{2} \sqrt{\frac{\hbar}{2}} C. \quad (1)$$

Since $\sum_m |c_m|^2 = 1$, we obtain

$$|c_0|^2 + |c_2|^2 + |c_{-2}|^2 = 1, \text{ so that}$$

$$\frac{\pi}{2} C^2 \left(1 + \frac{1}{4} + \frac{1}{4}\right) = 1. \text{ This means}$$

that $\frac{\pi}{2} C^2 \cdot \frac{3}{2} = 1$ and

$$C = \sqrt{\frac{4}{3\pi}} = \frac{2}{\sqrt{3\pi}}. \quad (2)$$

Substitution of (2) into (1) gives us

$$c_0 = \sqrt{\frac{2}{3}}, \quad c_2 = c_{-2} = \frac{1}{\sqrt{6}}, \text{ so}$$

that

$$P(m\hbar) = |c_m|^2 \text{ are,}$$

$$P(0) = |c_0|^2 = \frac{2}{3},$$

$$P(2\hbar) = P(-2\hbar) = |c_2|^2 = |c_{-2}|^2 = \frac{1}{6}.$$

The possible values of L_z are
 0 , $2\hbar$, and $-2\hbar$, and the
 corresponding probabilities are
 $\frac{2}{3}$, $\frac{1}{6}$, and $\frac{1}{6}$, respectively.

Clearly, $\sum_m P(m\hbar) = \frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 1$.

Let us compute

$$\langle \hat{L}_z \rangle = \sum_m m\hbar P(m\hbar)$$

(in general, $\langle \hat{A} \rangle = \sum_m a_m P(a_m)$).

We obtain,

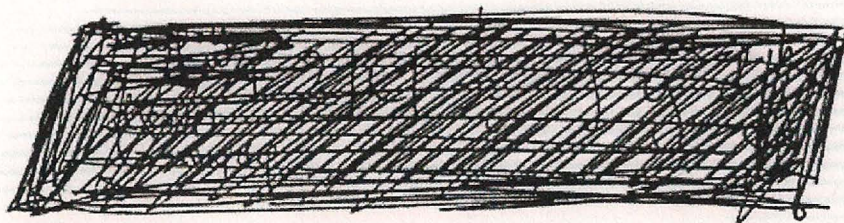
$$\begin{aligned} \langle \hat{L}_z \rangle &= \underbrace{0\hbar \frac{2}{3}}_{m=0} + \underbrace{2\hbar \frac{1}{6}}_{m=2} + \\ &\quad + \underbrace{(-2\hbar) \frac{1}{6}}_{m=-2} = 0. \end{aligned}$$

Let us try to calculate

$$\langle \hat{L}_z \rangle = \frac{\int_0^{2\pi} \psi^*(\varphi) \hat{L}_z \psi(\varphi) d\varphi}{\int_0^{2\pi} \psi^*(\varphi) \psi(\varphi) d\varphi}$$

$$\int_0^{2\pi} \psi^*(\varphi) \psi(\varphi) d\varphi = |C|^2 \int_0^{2\pi} \cos^4 \varphi d\varphi$$

$$\begin{aligned} \int_0^{2\pi} \psi^*(\varphi) \hat{L}_z \psi(\varphi) d\varphi &= |C|^2 \int_0^{2\pi} \cos^2 \varphi \frac{\hbar}{i} \frac{\partial}{\partial \varphi} \cos^2 \varphi d\varphi \\ &= -2|C|^2 \frac{\hbar}{i} \int_0^{2\pi} \cos^3 \varphi \sin \varphi d\varphi \end{aligned}$$



$= 0$, since

$$\begin{aligned} \int_0^{2\pi} \cos^3 \varphi \sin \varphi d\varphi &= - \int_0^{2\pi} \cos^3 \varphi d(\cos \varphi) = \\ &= - \left. \frac{\cos^4 \varphi}{4} \right|_0^{2\pi} = 0 \end{aligned}$$

Thus,

$$\langle \hat{L}_z \rangle = 0,$$

independent of the method used to calculate $\langle \hat{L}_z \rangle$.

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4. We know that

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \frac{\hbar}{i} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right),$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \frac{\hbar}{i} \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right),$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right).$$

Let us consider $\hat{L}_x \psi(r)$, where $\psi(r)$ is a spherically symmetric wave function.

We obtain,

$$\hat{L}_x \psi(r) = \frac{\hbar}{i} \left[y \frac{\partial \psi(r)}{\partial z} - z \frac{\partial \psi(r)}{\partial y} \right].$$

Since $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, we can write

$$\frac{\partial \psi(r)}{\partial z} = \frac{d\psi(r)}{dr} \frac{\partial r}{\partial z} = \frac{d\psi(r)}{dr}$$

$$\times \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$= \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \cancel{2z} \frac{d\psi(r)}{dr}$$

$$= \frac{z}{r} \frac{d\psi(r)}{dr}.$$

In other words,

$$\frac{\partial \psi(r)}{\partial z} = \frac{z}{r} \frac{d\psi(r)}{dr}.$$

Similarly,

$$\frac{\partial \psi(r)}{\partial y} = \frac{y}{r} \frac{d\psi(r)}{dr}.$$

Thus,

$$\begin{aligned}\hat{L}_x \psi(r) &= \frac{\hbar}{i} \left[y \frac{z}{r} \frac{d\psi(r)}{dr} - z \frac{y}{r} \frac{d\psi(r)}{dr} \right] \\ &= \frac{\hbar}{i} \left(\frac{yz - yz}{r} \right) \frac{d\psi(r)}{dr} = 0.\end{aligned}$$

We can repeat the analogous considerations for $\hat{L}_y \psi(r)$ and $\hat{L}_z \psi(r)$ and show that $\hat{L}_y \psi(r) = 0$ and $\hat{L}_z \psi(r) = 0$.

In other words,

$\hat{L}_x \psi(r) = 0 \cdot \psi(r)$, $\hat{L}_y \psi(r) = 0 \cdot \psi(r)$,
and $\hat{L}_z \psi(r) = 0 \cdot \psi(r)$, which means
that $\psi(r)$ is an eigenfunction of $\hat{L}_x$, $\hat{L}_y$,
and $\hat{L}_z$. The corresponding eigenvalues are 0.

Interestingly, the operator for the finite
rotation $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ $[\alpha_x, \alpha_y, \alpha_z$

are rotation angles corresponding to rotations about x , y , and z , respectively], which allows to rotate any function $f(x, y, z)$, can be given the form,

$$\hat{U}_{\vec{\alpha}} = e^{-i\vec{\alpha} \cdot \vec{\hat{L}} / \hbar},$$

where $\vec{\hat{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$.

Since $\psi(r)$ is spherically symmetric,

$$\hat{U}_{\vec{\alpha}} \psi(r) = \psi(r) \text{ for}$$

every $\vec{\alpha}$. This means that

$$e^{-i\vec{\alpha} \cdot \vec{\hat{L}} / \hbar} \psi(r) = \psi(r),$$

for all $\vec{\alpha}$'s, which implies that

$$\hat{L}_{x(y,z)} \psi(r) = 0.$$

Q.E.D.