CEM 991 Fall 2021

Assignment No. 6 (25 pts.) Due Date: November 22, 2021 (Monday)

1. (4 pts.) Use the following representation of the Dirac delta function [cf. E. Merzbacher, *Quantum Mechanics*, 3rd edition (Wiley, New York, 1998), Appendix, pp. 630-634]:

$$\delta(y) = \lim_{n\to\infty} \frac{\sin ny}{\pi y},$$

to show that

$$\int_{-\infty}^{+\infty} e^{ikx} dx = 2\pi\delta(k).$$

Hint: $\displaystyle\int_{-\infty}^{+\infty} e^{ikx} dx = \lim_{a\to\infty} \int_{-a}^{+a} e^{ikx} dx.$

2. (8 pts.) Let $\delta(x)$ be the Dirac delta function. Show that

   (a) $\delta(x) = \delta(-x)$.

   (b) $x\delta(x) = 0$.

   (c) $\delta(ax) = a^{-1}\delta(x)$, $a > 0$.

   (d) $\displaystyle\int_{-\infty}^{\infty} \delta(a - x)\delta(x - b)\, dx = \delta(a - b)$.

   Recall that the Dirac delta function is not an ordinary function. The meaning of each of the above equations is that the corresponding left- and right-hand sides, when used as multiplying factors under an integral sign, lead to the same result. For example, in order to prove (a), i.e., $\delta(x) = \delta(-x)$, you should prove that $\displaystyle\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(x)\delta(-x)dx$ for any function $f(x)$.

3. (8 pts.) The z component of the angular momentum operator $\hat{\mathbf{L}}$ is represented by the operator

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi},$$

where ϕ is a polar angle. The normalized eigenfunctions of $\hat{L}_z$ are

$$\phi_m = \frac{1}{\sqrt{2\pi}} e^{im\phi},$$

and the corresponding eigenvalues are $m\hbar$, where $m = 0, \pm1, \pm2, \ldots$. Consider a state described by the wave function

$$\psi(\phi) = C\cos^2\phi, \tag{1}$$

where C is a constant. Calculate the possible values $m\hbar$ of L_z and their probabilities $P(m\hbar)$ in a state described by the wave function $\psi(\phi)$, Eq. (1). Calculate the expectation value

$$\langle \hat{L}_z \rangle = \frac{\int_0^{2\pi} \psi^*(\phi)\hat{L}_z\psi(\phi)d\phi}{\int_0^{2\pi} \psi^*(\phi)\psi(\phi)d\phi}$$

and compare the result with the expectation value of $\hat{L}_z$ defined as

$$\langle \hat{L}_z \rangle = \sum_m m\hbar P(m\hbar).$$

Hint: Use the equations $\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$ and $\cos\alpha = \frac{1}{2}(e^{i\alpha}+e^{-i\alpha})$.

4. (5 pts.) Consider a one-particle system in a state described by a spherically-symmetric wave function $\psi = \psi(r)$ (r is the distance between particle and the origin). Show that ψ is a simultaneous eigenfunction of all three angular momentum operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$, in spite of the fact that $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ do not commute. Find the corresponding eigenvalues. *This example demonstrates that non-commuting operators may have common eigenstates.*